Research Statement

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I am interested in the general realm of intersection between representation theory, algebraic geometry, and mathematical physics. More specifically, I am concerned with questions that arise in the study of monstrous moonshine. Important progress has been made in the last 20 years, starting with Borcherds's celebrated proof of the monstrous moonshine conjecture.

My primary goal in the near future is the proof the generalized moonshine conjecture, proposed by Norton in 1986. This conjecture was discovered through numerical experimentation, but it hints at deep structures in its assertion of a relationship between the representation theory of the monster simple group and Fourier coefficients of genus zero modular functions. I have recently submitted work that reduces this conjecture to a pair of hypotheses concerning groups acting on vertex algebras and Lie algebras, and I intend to resolve those hypotheses.

Generalized moonshine - Motivation and previous results

Monstrous moonshine arose in the 1970s when several mathematicians observed numerical relationships between the monster group $\mathbb{M}$ and modular functions invariant under certain discrete subgroups of $PSL_2(\mathbb{R})$, therefore suggesting a relationship between the fields of finite group theory and modular functions, which were until then thought to be unrelated. Conway and Norton showed in their seminal paper *Monstrous Moonshine* that they could combine entries from the character table of the monster group $\mathbb{M}$ to generate power series, called McKay-Thompson series, that agreed with genus zero modular functions up to the first several terms. Based on this, they formulated the moonshine conjectures [CN79], which propose the existence of an infinite dimensional graded representation $V^\natural$ of the monster, such that for any $g \in \mathbb{M}$, the McKay-Thompson series

$$T_g(\tau) = \sum_{n \geq -1} Tr(g|V_n^\natural)q^n$$

forms the cusp expansion of a modular function that is invariant under a genus zero group of Möbius transformations. The genus zero property is the condition that if $T_g(\tau_1) = T_g(\tau_2)$ for any $\tau_1, \tau_2 \in \mathfrak{H}$, there exists an element of $PSL_2(\mathbb{R})$ that fixes T_g, and takes τ_1 to τ_2. Frenkel, Lepowsky, and Meurman [FLM88] constructed $V^\natural$ as a vertex algebra, called the moonshine module, equipped with a faithful action of the monster, and Borcherds [B92] proved that the graded traces arising from the action were the modular functions proposed by Conway and Norton.

Queen's dissertation [Q81] gave numerical evidence that moonshine-like behavior was not limited to the monster simple group, but that other sporadic groups also gave rise to modular functions in similar ways. By combining this evidence with his own computations, Norton [N87, N01] formulated a generalized moonshine conjecture that remains open today. This conjecture asserts the existence of a generalized character, which associates to any ordered pair of commuting elements of the monster a genus zero modular function, defined up to multiplication by a nonzero constant. These functions must satisfy compatibility conditions with respect to a canonical set-theoretic action of $SL_2(\mathbb{Z})$ on the abelian group generated by the two elements. In the language of algebraic geometry, such a generalized character is a multivalued function on the moduli stack of elliptic curves equipped with monster torsors.

Both the construction of the moonshine module and the proof of the moonshine conjectures used techniques from the theory of infinite dimensional Lie algebras motivated by theoretical physics. This suggests that string theory, or a part of it, acts as a bridge between finite group theory and modular functions. The functions Norton computed also have a physical interpretation in terms of conformal field theory. In particular, they are conjectured [DGH88] to be the graded traces of centralizing elements acting on the irreducible twisted modules of the moonshine vertex algebra $V^\natural$. Dong, Li, and Mason [DLM00] showed that $V^\natural$ has exactly one isomorphism class of irreducible twisted module for each element of the monster. Furthermore, they showed, by reducing to Borcherds' theorem, that if the commuting pair generates a cyclic group, the traces of centralizing elements obey the $SL_2(\mathbb{Z})$-compatibility in Norton's conjecture up to a constant factor, and are therefore genus zero functions.

My dissertation and later work concerns the non-cyclic case. In the following sections, I will describe my results and plans for the next few years, together with connections to other areas of mathematics that are relevant to their solution.

Program of proof

Borcherds's proof of the monstrous moonshine conjecture [B92] can be roughly split into the following steps:

1. Construct a vertex operator algebra with a faithful action of the monster (done by Frenkel, Lepowsky, and Meurman).
2. Use a functor from vertex algebras to Lie algebras to construct a Lie algebra with a monster action, and use the no-ghost theorem to determine its root multiplicities.
3. Construct an abstract generalized Kac-Moody Lie algebra with an automorphic denominator formula. This will have precisely determined simple roots and root multiplicites.
4. By comparing the root multiplicities, show that the two Lie algebras are isomorphic. This allows one to combine the group action with knowledge of the simple roots.

5. From a generalized Kac-Moody algebra with a homogeneous action of the monster, one obtains twisted denominator identities that relate characters of elements acting on the root spaces.

6. The resulting recursion relations on characters are satisfied by the proposed genus zero functions (shown in unpublished work of Koike) and strong enough that the functions are uniquely determined by the first several terms. The conjecture is proved by checking that these terms match the proposed ones.

My goal is to extend this recipe to deal with the objects that show up in generalized moonshine, such as twisted modules. This extension was already accomplished by Hoehn for the special case when the twisting element lies in conjugacy class 2A [H03], and I have based my strategy on the architecture of his proof. The general case is substantially more complicated than the 2A case, since it is unreasonable to do explicit twisted module constructions for each of the remaining 192 conjugacy classes, and because the relevant character tables of centralizers are not known in all cases. The outline is approximately the following:

1. For each element g of the monster simple group, construct a generalized vertex algebra as a direct sum of irreducible g^i-twisted modules, for $0 \leq i < |g|$.

2. Use a functor from vertex algebras to Lie algebras to construct a Lie algebra with a monster action, and use a suitable version of the no-ghost theorem to determine its root multiplicities.

3. For each element g of the monster, construct an abstract generalized Kac-Moody Lie algebra with an automorphic denominator formula that involves the McKay-Thompson series $T_g(\tau)$.

4. For each element of the monster, show that the root multiplicities of the corresponding Lie algebras are equal, and that the Lie algebras are therefore isomorphic.

5. From a generalized Kac-Moody algebra with a homogeneous action of a finite group, one obtains twisted denominator identities that relate characters of elements acting on the root spaces.

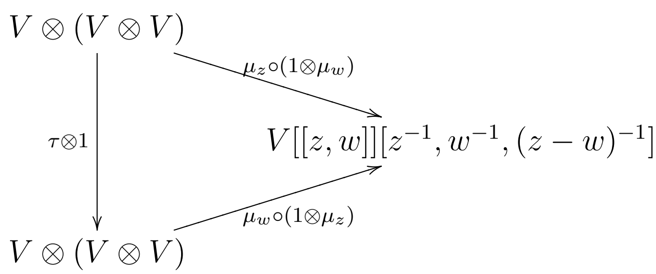
6. Show that the recursion relations on characters imply they are genus zero functions when possible (i.e., when the McKay-Thompson series of the twisting element has a pole at zero).

7. Show that pseudo-traces on weak twisted modules span a space of genus 1 functions with good modular properties. Combined with the recursion relations, this should yield $SL_2(\mathbb{Z})$-compatibility together with the remaining genus zero results.

Step 2 requires minimal alteration from Borcherds's theorem, but each of the remaining steps is highly nontrivial. In [C08], I completed step 6, and in [C09], I completed steps 3 and 5. In my doctoral dissertation, I made some partial progress toward steps 1 and 4. I will state precise versions of the theorems and conjectures after some background discussion.

Orbifold conformal blocks

The first step in my attack on generalized moonshine is a construction arising from the theory of vertex algebras. A vertex algebra is a vector space V equipped with a multiplication map $\mu_z : V \otimes V \to V((z))$, a unit $1 \in V$, and an endomorphism $T : V \to V$. These data are required to satisfy compatibility axioms, the most crucial of which is called "locality" and encodes a sort of commutativity and associativity: If $\tau : V \otimes V \to V \otimes V$ is the switch map defined by $A \otimes B \mapsto B \otimes A$, then the following diagram commutes:



Geometrically speaking, we can consider a vector bundle on a small disc with fiber V, and the multiplication map sends two sections of the fiber over a point to a meromorphic section defined in a small neighborhood away from that point. There is a similar picture for modules over a vertex algebra, and there is a modified version involving a branched cover of the disc that applies to twisted modules. Work of Frenkel, Ben-Zvi, and Szczesny [FBZ04], [FS04] has made this heuristic picture precise, and they have given a rigorous definition of the physical notion of conformal blocks on an algebraic curve associated to twisted module insertions. Conformal blocks can be interpreted as either maps of representations of a Lie algebra constructed from a vertex algebra, or as linear functionals satisfying an analytic continuation condition. By studying the variation of conformal blocks with respect to deformations of the curve or movement of marked points, one can gain insight on the behavior of correlation functions and intertwining operators.

I propose to apply this machinery to studying the twisted modules that are conjectured to be the key objects in generalized moonshine. In my dissertation, I gave some arguments toward a proof of the following:

Theorem: Let V be a holomorphic C_2-cofinite vertex operator algebra, with an action of a finite group G by conformal automorphisms. Let $C \to \mathbb{P}^1$ be a G-cover equipped with marked points $q_1, \ldots, q_n \in C$ with monodromy $g_1, \ldots, g_n$, such that their images $p_1, \ldots, p_n \in \mathbb{P}^1$ contain the branch locus. The space of conformal blocks with irreducible g_i-twisted modules inserted along orbits of q_i is one-dimensional.

This is a twisted module analogue of the main theorem in [NT05]. Twisted conformal blocks in higher genera and conformal blocks in positive characteristic are essentially completely unexplored territory, but I expect to study them later as a peripheral project.

In order to complete the analogue of step 1 in my attack on generalized moonshine, I need to construct a generalized vertex algebra V_g for each $g \in \mathbb{M}$ as a direct sum of g^i-twisted modules, such that V_g has an action of a central extension of $C_{\mathbb{M}}(g)$ by automorphisms. Generalized vertex algebras as defined by Dong and Lepowsky are quite similar to vertex algebras, but the multiplication map has the form $V \otimes V \to V((z^{1/N}))$, and the locality axiom involves additional correction terms arising from nontrivial monodromy. In order to construct such an object from twisted modules, I need a multiplication map to be defined on the modules, and the above theorem on conformal blocks implies that the space of such maps that is compatible with the twisted module structure is one dimensional. To show that the direct sum satisfies the locality axiom, I need a calculation using an obstruction theory that is the subject of the next section.

Vertex algebras in a braided tensor category

In the previous paragraph, I mentioned one generalization of the notion of vertex algebra, and there are several more in the literature. If we consider vertex algebras to be commutative rings with singularities in the multiplication, many of them can be viewed as straightforward alterations of this idea. Dong and Lepowsky [DL93] first constructed graded objects, for which the usual singular commutativity has some alteration by roots of unity. The field algebras of [BK03] and the nonlocal algebras [Li05] remove the singular commutativity entirely to yield a singular associative ring. Treatments in [EK00], [FR97] and [B01] use a notion of braided commutativity with respect to an R-matrix. The main motivation for the braided examples seems to be constructing certain representations of quantum affine algebras, such as that in [FJ88]. Finally, Beilinson and Drinfeld [BD04] have produced several equivalent geometric notions of chiral algebra that involve D-modules on a curve, divisors in a fibered category, or crystals on diagrams of schemes, and they have set up axiomatics for notions like commutative algebras in symmetric pseudo-tensor categories. In the $\mathbb{G}_a$-equivariant setting on the affine line, these chiral algebras are equivalent to vertex algebras in the classical sense.

It has been suggested in [B01] that one could add the braiding at the outset, and consider singular commutative rings in a braided tensor category. The idea of working in the braided universe from the beginning has appeared outside vertex algebra theory, in Majid and Lyubashenko's work on braided groups and transmutation (e.g., in [LM94]). They proved a braided version of Tannakian reconstruction, where instead of automorphisms of a forgetful functor forming a pro-algebraic group, certain natural endomorphisms on the identity functor on a braided tensor category form a Hopf algebra in that category, assuming certain representability conditions. Much like the classical Tannakian setting, these Hopf algebras have the property that the original braided tensor category is equivalent to the module category of the Hopf algebra. However, they tend to have properties that appear nicer than the Hopf algebras in vector spaces obtained from automorphisms of a forgetful functor. In particular, for quantum enveloping algebras and Drinfeld doubles of finite groups, they are commutative, cocommutative, and self-dual.

A physical motivation for studying braided structures is that they naturally form when one considers configurations of codimension 2 objects, such as in knot theory, and the field insertions at points in a complex curve in conformal field theory are a sufficiently simple case that one can reasonably hope to make rigorous constructions. My primary motivations for considering vertex algebras in a braided tensor category are that some existing constructions may be recast to appear more natural, much like the transmutation situation, and that conceptual simplifications could lead to new constructions that are not obvious in the symmetric setting.

The actual definition involves only minor structural alterations to Borcherds's abstract categorical definition [B01]. The main point is to break the symmetry on inputs where it is present, so we replace categories of finite sets with categories of objects that come with ordering. By MacLane's coherence theorem, we may assume the braided category has a strictified monoidal structure, so we use the category of finite linearly ordered sets, and impose braid group equivariance by hand.

There is also a translation to the geometric language of chiral algebras, introduced by Beilinson and Drinfeld. They give several equivalent definitions of chiral algebra, including:

1. A Lie algebra in D_X-modules (under the chiral pseudotensor structure)
2. A D_{X^n}-module for each non-negative integer n, together with a gluing rule along diagonal embeddings and a factorization isomorphism on open complements.
3. A quasicoherent sheaf on a "space" of effective Cartier divisors on X.

While the last definition is somehow unalterably symmetric, the symmetry in the first two definitions can be broken by introducing orderings on index sets as before. In order to admit chiral versions of the vertex algebra constructions to come, we will need to employ monodromic D-modules. Fortunately, the foundations for these objects were laid out in [BB93]. This extra generality also allows us to translate the traditional language of modules over vertex algebras with nonintegral conformal weights to the chiral setting.

There is an important special case of the general definition, when the underlying abelian category is that of complex vector spaces graded by an abelian group H. Joyal and Street [JS86] showed that braided tensor structures on this category are classified up to braided tensor equivalence by $\mathbb{C}^\times$-valued quadratic forms Q on H, so I will call these categories $Vect^H_Q$ (although in practice one chooses explicit commutors and associators). Vertex algebras in $Vect^H_Q$ have already appeared in the literature in a disguised form. When H is finite and Q takes values in $\mu_n(\mathbb{C})$, they are called abelian intertwining algebras of level n, in the sense of [DL93]. If Q is representable with a trivial associator, these are the generalized vertex algebras defined in [BK06]. If Q is the constant form 1, then the braiding is trivial, and these are just vertex algebras equipped with an action of the commutative diagonalizable group $D(H)$ (defined in [DG64]). This group is characterized via Tannaka-Krein duality by the property that its category of finite dimensional representations is the full subcategory of finite dimensional objects in $Vect^H_1$.

The standard example of vertex algebras in $Vect^H_Q$ arises from a lattice L whose quadratic form is not necessarily integral. Dong and Lepowsky [DL93] constructed these lattice vertex algebras in the case that L has a finite index even integral sublattice, and Bakalov-Kac [BK06] gave a construction corresponding to any subgroup of $L \otimes \mathbb{C}$. Since generalized lattice algebras have a direct construction, we don't need to calculate intertwining operators between modules.

A richer example arises from taking a sum of irreducible g^i-twisted modules of the monster vertex algebra for some $g \in \mathbb{M}$. Twisted modules can be viewed as ordinary modules if we suitably expand our category, and to give such a sum a vertex algebra structure requires one to produce a multiplication map that satisfies singular commutativity and associativity. Given compatible one-dimensional spaces of intertwining operators (implied by the conformal blocks theorem above), one obtains a canonical obstruction class in $H^4(K(H,2), \mathbb{C}^\times)$, for H an abelian group such that the twisted modules lie in $Vect^H$, and the conformal weight of any homogeneous component lies in a single coset of $\mathbb{Z}$. The class is then completely determined by the conformal weights. Since this cohomology group also classifies braided structures on the category of H-graded vector spaces, we can trivialize the class by altering the braided structure in which the twisted modules live.

My attack on generalized moonshine needs only vertex algebras in H-graded vector spaces, but the general framework has additional interesting potential examples. For example, one can try to construct a vertex algebra structure on the direct sum of all irreducible g-twisted $V^\natural$-modules for g ranging over all elements of the monster rather than restricting to a cyclic group. This is a priori a noncommutative structure, since the monster is nonabelian. However, if we place the twisted modules in a suitable braided tensor category, the structure transmutes to a commutative one. In this case, the category in question is that of $D^\omega(G)$-modules, where $\omega \in H^3(G, \mathbb{C}^\times)$ and $D^\omega(G)$ is the ω twisted double of G [DPR90]. As in the cyclic case, the conformal block theorem yields one dimensional spaces of intertwining operators. However, the obstruction theory involves the cohomology of the two-fold delooping of a space that is more nontrivial than an abelian group. In particular, it is a free loop space of BG^ω, where G^ω is a 2-group, or categorical group. I intend to investigate further the role of two-fold loop spaces in these structures.

A second important potential example is given by a vacuum representation of a quantum affine algebra. If I can recast the constructions in [FJ88] and [EK00] as objects living in a suitable braided tensor category, it may be appear more natural, and it will have the advantage of admitting a theory of modules as split square zero extensions. Assuming this works, I conjecture that all integrable modules with a fixed central charge admit module structures under the integrable quotient at positive integral level, analogous to the theorem of [FZ92] in the unbraided case.

The chiral algebras analogous to vertex algebras in $Vect_Q^H$ have a concrete description using monodromic D_X-modules, and the generalized lattice algebras mentioned before are a natural example here. The chiral version of their construction is an extension of that in [BD04], section 3.10, where we now remove the condition that a certain bilinear form be integer-valued, and consider line bundles twisted by a flat gerbe.

Chiral homology of a chiral algebra is a derived version of the space of coinvariants for modules of a vertex algebra, and in some special cases, can be used to produce interesting correlation functions. Beilinson and Drinfeld [BD04] have a computation of the chiral homology of lattice chiral algebras for all genera, using a Fourier-Mukai transform. I am currently working on an extension of this result to generalized lattice algebras, including setting out axiomatics for chiral homology in the braided context.

Borcherds products, Lie algebras, and genus zero functions

This part of the attack on the generalized moonshine conjecture is already completed, and it concerns steps 3,5, and 6 in the outline given above. I will give a short summary of results to add context.

Theorem: Let g be an element of the monster simple group $\mathbb{M}$, and let N be the level of the McKay-Thompson series $T_g(\tau)$. Then the function $T_g(\sigma) - T_g(\tau)$ on $\mathfrak{H} \times \mathfrak{H}$ has an infinite product expansion at the cusp $(i\infty, 0)$ of the form

$$p^{-1} \prod_{m \in \mathbb{Z}_{>0}, n \in \frac{1}{N}\mathbb{Z}} (1 - p^m q^n)^{c_{m,Nn}(mn)}$$

where $p = e^{2\pi i \sigma}$, $q = e^{2\pi i \tau}$, and each $c_{m,Nn}(mn)$ is a non-negative integer coefficient of a vector-valued modular function assembled from the expansions of $T_{g^i}(\tau)$ at various cusps.

The above theorem and the following were proved in [C09].

Theorem: For each $g \in \mathbb{M}$, there exists a unique generalized Kac-Moody Lie algebra

$\mathfrak{m}_g$ whose Weyl-Kac-Borcherds denominator identity has the form:

$$T_g(\sigma) - T_g(-1/\tau) = p^{-1} \prod_{m \in \mathbb{Z}_{>0}, n \in \frac{1}{N}\mathbb{Z}} (1 - p^m q^n)^{c_{m,Nn}(mn)}$$

This Lie algebra is graded by $\mathbb{Z} \times \frac{1}{N}\mathbb{Z}$, with the degree $(0,0)$ part given by the torus $\mathbb{C}^2$. For $(m,n) \neq (0,0)$ the degree (m,n) homogeneous component has dimension given by the coefficient $c_{m,Nn}(mn)$ of a certain vector-valued modular form.

The next theorem uses some technical language introduced in [C08]. In that paper, I developed a notion of Hecke-monic function on a moduli space of elliptic curves with G-torsors, and I showed that weakly Hecke-monic functions with poles at infinity yield genus zero functions on the upper half plane.

Theorem: Assume some central extension of $C_{\mathbb{M}}(g)$ acts on the Lie algebra $\mathfrak{m}_g$ by homogeneous Lie algebra automorphisms, and assume there are isomorphisms of representations between the degree (m,n) part of $\mathfrak{m}_g$ and the subspace of the irreducible g^m-twisted $V^\natural$-module on which g acts by $e^{2\pi i n}$ and the conformal operator L_0 acts by $1+mn$. Then the twisted denominator identity implies the characters of automorphisms of twisted modules are weakly Hecke-monic. If g is Fricke (i.e., if $T_g(\tau)$ has a pole at zero), then these characters are genus zero functions of finite level.

The following theorem uses some technical language from [M04] and [DLM00]. The assumption is a "least common generalization" of Theorem 5.5 in [M04] and Theroem 10.4 in [DLM00]

Theorem: Assume the space $C_1(g,h)$ of genus 1 functions associated to $V^\natural$ is spanned by pseudo-traces of lifts of h on g-twisted Verma modules, and assume the subspace of series with no logarithmic terms is spanned by traces on twisted modules. Then under the assumptions of the previous theorem, the generalized moonshine conjecture is true.

Combined with the work described in the previous sections, these results reduce the generalized moonshine conjecture to a root multiplicity comparison (step 4) and a question about pseudo-traces (step 7).

Comparison

The main results of [C09] give constructions of Lie algebras by generators and relations, but to endow them with natural actions of groups, we have to show that they are abstractly isomorphic to the Lie algebras arising from the twisted module constructions. Since both algebras are generalized Kac-Moody algebras, it suffices to show that the root multiplicities are the same.

The no-ghost theorem identifies the root spaces with certain eigenspaces of twisted modules of the monster vertex algebra, so we need to understand the dimension of these eigenspaces. This requires us to understand both the size of a twisted module, and the action of a cyclic group on them. Unfortunately, there is little precise information in the literature on either problem. For each $g \in \mathbb{M}$, a theorem of [DLM00] gives the character of g^j acting on an irreducible g^i-twisted module up to a nonzero constant. In particular, the character of g^0 is the graded dimension, and it is some constant times $T_{g^i}(-1/\tau)$. Until now, the constants in question have been known only when g is the identity or an element of order two. By using the conformal block theorem in step 1, I can determine the dimension of all irreducible twisted modules of $V^\natural$:

Theorem: For any $g \in \mathbb{M}$, $Tr(q^{L_0-1}|V^\natural(g)) = T_g(-1/\tau)$.

This theorem uses a combination of tricks from the theory of generalized Kac-Moody algebras and fusion of twisted modules for the Leech lattice vertex algebra. One immediate consequence of this theorem is that one can identify the Lie algebras corresponding to elements of prime order. The question is substantially more subtle for the characters of elements of composite order, but I believe I have a strategy for resolving this.

Pseudo-traces

The final step in my strategy for solving the generalized moonshine conjecture is to prove a spanning theorem for a space of correlation functions introduced by Dong, Li, and Mason. This work is in its preliminary stages.

There is a theorem of Zhu [Z96], asserting that if V is a rational C_2-cofinite vertex operator algebra, then the characters of its modules form functions on the complex upper half plane, such that the space spanned by the characters is taken to itself under the action of $SL_2(\mathbb{Z})$. Miyamoto [M04] proved a generalization of this theorem that removes the rationality assumption, but to compensate, one expands the scope from characters to pseudo-traces. Dong, Li, and Mason [DLM00] generalized Zhu's theorem in a different direction, studying characters of twisted modules. They introduced a space $C_1(g,h)$ of genus 1 functions associated to a pair of commuting automorphisms of V, and showed that this space satisfied a modular invariance condition. Furthermore, they showed that if V is g-rational and C_2-cofinite, then the space is spanned by the character of h acting on g-twisted modules of V. $V^\natural$ is C_2-cofinite, and if we assume it is g-rational for all $g \in \mathbb{M}$, we automatically get modular invariance and $SL_2(\mathbb{Z})$-compatibility up to a constant for these characters, solving a large part of the generalized moonshine conjecture. Unfortunately, g-rationality seems to be extremely difficult to prove. We propose to avoid this issue with the following:

Conjecture: For any pair of commuting elements $g, h \in \mathbb{M}$, the space $C_1(g,h)$ corresponding to $V^\natural$ is spanned by pseudo-traces of lifts of h on g-twisted Verma modules. Furthermore, the subspace of series with no logarithmic terms is spanned by traces on twisted modules.

I showed in [C09] that the generalized moonshine conjecture is implied by combining this conjecture with twisted denominator identities. Many of the steps in Miyamoto's proof of the untwisted case can be converted with minimal effort to the twisted module setting that we need, but there are still significant technical obstructions.

Chiral algebras and logarithmic structures

In the proof of the main theorem on conformal blocks in [NT05], there is a key step involving a sheaf of Lie algebras on a semistable family of genus zero curves. This sheaf is constructed by an ad hoc technique involving powers of the relative dualizing sheaf, and a straightforward generalization to higher genera would require that a vertex algebra be spanned by quasi-primary vectors. However, there exists a framework in which the Lie algebras are quite natural, given by combining the coordinate-free conformal block construction of [FBZ04] with the theory of logarithmic structures introduced by Fontaine, Illusie, and Kato [K89].

Logarithmic geometry was developed to deal with questions of semistable degeneration and crystalline cohomology in positive characteristic, but it also yields a powerful toolset for understanding compactification of moduli in the complex universe. In our setting, we wish to study how sheaves of conformal blocks on a family of smooth curves behave as one approaches the boundary of the moduli space. The Frenkel-Ben-Zvi definition of conformal blocks uses the smooth structure on a curve to construct a chiral algebra whose fiber is the vertex algebra in question, but the nodal curves that appear in the Deligne-Mumford compactification of moduli are not smooth, so they do not have well-behaved coordinate bundles. This problem can be fixed, because nodal curves admit canonical logarithmically smooth logarithmic structures, and this allows one to construct chiral algebras using a logarithmic Gelfand-Kazhdan equivalence. One obtains a sheaf of Lie algebras on the compactified space which, in the nodal genus zero case, agrees with the dualizing sheaf construction in [NT05]. By logarithmic Beilinson-Bernstein localization, the twisted D-module of conformal blocks extends to the compactified moduli space, with logarithmic singularities at the boundary. This construction also works for twisted modules, because results of Olsson [O07] produce canonical log structures on twisted curves.

I have some preliminary results comparing conformal blocks on any nodal curve with conformal blocks on the normalization. In the genus zero untwisted case, this is essentially done in [NT05], but my results seem to extend to all genera and with twistings. I believe this may yield finiteness theorems for conformal blocks in all genera for arbitrary C_2-cofinite vertex operator algebras (i.e., without semisimplicity assumptions), and it may yield a new proof of the Verlinde formula [H08].

Elliptic cohomology and conformal field theories

Other future research will involve examining the relationship between vertex algebras and elliptic cohomology. There are many hints that elliptic cohomology is a "shadow" of conformal field theory [ST04], and the generalized moonshine conjecture is a precise manifestation of this. The $SL_2(\mathbb{Z})$ compatibility condition naturally makes the collection of characters a multivalued function on the moduli stack of elliptic curves equipped with monster torsors, and the condition on representations of centralizers places the cusp expansions in twisted S^1-equivariant K-theory of the loop space of $B\mathbb{M}$. Equivariant K-theory is a degeneration of elliptic cohomology to the Tate curve, and by some form of q-expansion principle, still conjectural in this setting, the generalized moonshine conjecture is an assertion that $V^\natural$ together with its twisted modules represents a monster-equivariant elliptic cocycle, twisted by some element of $H^4(\mathbb{M}, \mathbb{Z})$, whose isomorphism type is not known. This is interesting in part because elliptic cocycles are quite difficult to construct, and it seems to be unclear what algebraic information needs to be written down to capture the data of an elliptic cocycle. Some work along these lines has been done in [G07].

As a partial converse to the above question, the generalized character theory of Hopkins, Kuhn, and Ravenel [HKR00] gives a function on the pairs of commuting elements of the monster for any monster-equivariant elliptic cocycle, because the Barsotti-Tate groups of elliptic curves have height 2. Furthermore, if we assume $V^\natural$ fits into this framework, there is a formal exponential operation, arising from weighted summation over finite flat subgroup schemes of elliptic curves, that produces the positive parts of the monstrous Lie algebras $\mathfrak{m}_g$ described earlier. The highly degenerate nature of the denominator identites suggests that genus zero functions are distinguished by a "self-exponential" property, analogous to the formulas $Sym(\mathbb{C}[1]) - \mathbb{C} = \mathbb{C}[1]$ and $e^0 - 1 = 0$ in complex K-theory and cohomology, respectively. One long term goal is to establish a more solid connection between this exponential operation, second quantization (as developed in [DF09] for studying three dimensional quantum gravity), and passage to the space of physical states via BRST cohomology.

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