The geometry of Markov traces

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Abstract. We give a geometric interpretation of the Jones-Ocneanu trace on the Hecke algebra, using the equivariant cohomology of sheaves of SL_n. This construction makes sense for all simple groups, so we obtain a generalization of the Jones-Ocneanu trace to Hecke algebras of other types. We show that this trace coincides (up to choice of normalization) with that defined by Gomi, and give a short geometric proof of Gomi's expansion of this trace in terms of characters.

Based on our proof, we also prove that certain simple perverse sheaves on G are equivariantly formal for the conjugation action of B, or equivalently, that the Hochschild homology of any Soergel bimodule is free, as the authors had previously conjectured.

This construction is also closely tied to knot homology. This interpretation of the Jones-Ocneanu trace is a more elementary manifestation of the geometric construction of HOMFLYPT homology by the authors [WW].

1. **Introduction.** In his original paper on the relationship between Hecke algebras and the HOMFLYPT polynomial [Jon87], Jones asks whether the Jones-Ocneanu trace, which he constructs by an inductive procedure, has some direct interpretation in terms of the Kazhdan-Lusztig basis of [KL79].

In this paper, we describe a relationship between this trace on a Hecke algebra and the geometry of the corresponding Lie groups, which passes through the "Hecke category" of $B \times B$-equivariant perverse sheaves on G. Since the Kazhdan-Lusztig basis can also be defined in terms of this category, this provides at least one answer to Jones's question:

Theorem. *The Jones-Ocneanu trace applied to a Kazhdan-Lusztig basis element* C_w *is the* B_Δ*-equivariant mixed Poincaré polynomial of the corresponding intersection cohomology sheaf* $\mathbf{IC}_w$*. Equivalently, it is the bigraded dimension of the Hochschild homology of an indecomposible Soergel bimodule* S_w*.*

Here, B_Δ denotes the upper triangular matrices acting by conjugation on G. The equivalence of these statements follows from [WW08, Theorems 1.2 & 1.4].

Interestingly, this geometric construction works in all types, not just SL_n, providing a natural trace on the Hecke algebra for any Dynkin diagram. Generalizing the Jones-Ocneanu trace to other types has been an active avenue of research for many years. Traces satisfying a Markov-type condition on Hecke algebras of classical type were classified by Geck and Geck-Lambropoulou [Gec98, GL97], but such traces form an infinite dimensional vector space. Gomi [Gom06] constructed a special trace on the Hecke algebra of any type (even non-crystallographic) which satisfies a strong Markov condition by giving an explicit formula as a sum of the characters of irreducible representations.

Using the connection between the Hecke category on G and the theory of character sheaves, we give a simple geometric recipe for expanding our trace in terms of characters (Corollary 13). This shows, using no combinatorics, that our trace coincides with that defined by Gomi, and gives a geometric interpretation of Gomi's formula.

In the course of our proof, we must show the following result, which is of some independent interest:

Theorem. *Any simple $B \times B$ equivariant perverse sheaf on G is B_Δ-equivariantly formal. Equivalently, the Hochschild homology of any Soergel bimodule is free.*

This statement was previously proved by Rasmussen [Ras] using an inductive calculation (similar to Jones's construction of the trace) in type A, but we give an independent, purely geometric proof using Lusztig's work on parabolic induction and restriction functors for character sheaves.

Just as this trace is intimately tied to the HOMFLYPT polynomial as described in [Jon87, FYH$^+$85], its categorification is connected to the triply graded homology of Khovanov and Rozansky [Kho07, KR08, Ras]. The main theorem of this paper is essentially equivalent to the fact that this knot homology is, in fact, a knot invariant categorifying the HOMFLYPT polynomial, but we provide a description of the categorified trace on the Hecke algebra from first principles, separate from knot theory. The authors relate this description to knot theory in a separate paper [WW], which describes a geometric construction of colored HOMFLYPT homology, whose construction had been proposed by Mackaay, Stosic and Vaz [MSV].

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2. **Hecke algebras.** Let Γ be a Dynkin diagram. Attached to this Dynkin diagram, we have:

- An Artin braid group $\mathcal{B}(\Gamma)$, which is generated by symbols σ_i for $i \in \Gamma$, with the relations
$$\underbrace{\sigma_i\sigma_j\cdots}_{m_{ij}\text{ terms}} = \underbrace{\sigma_j\sigma_i\cdots}_{m_{ij}\text{ terms}}$$
where m_{ij} is the multiplicity of the line connecting i and j in the Dynkin diagram.
- A Coxeter group $W(\Gamma)$ obtained as the quotient of $\mathcal{B}(\Gamma)$ by the relation $\sigma_i^2 = 1$, which is the Weyl group of the associated complex simple Lie algebra. We use s_i to denote the image of σ_i in $W(\Gamma)$.
- A Hecke algebra $\mathbf{H}(\Gamma)$, which is the quotient of the group algebra of the braid group $\mathcal{B}(\Gamma)$ over $\mathbb{Z}[q^{1/2}, q^{-1/2}]$ by the relations
$$(\sigma_i - q^{1/2})(\sigma_i + q^{-1/2}) = 0.$$
This is a "deformation" of the relation $\sigma_i^2 = 1$, and the resulting algebra is a flat deformation of the group algebra of $W(\Gamma)$. In fact, there is a "standard" basis σ_w of $\mathbf{H}(\Gamma)$ where the basis elements are labeled by $w \in W(\Gamma)$, which limits to the standard basis on the group algebra.

Obviously, if we have an inclusion of Dynkin diagrams $\Gamma' \to \Gamma$, then there is an induced map for each of the algebraic objects listed above. In particular, for each infinite series, we have a tower of inclusions.

We first consider the case of the A_n Dynkin diagrams. We let $\mathbf{H}(A_n) = \mathbf{H}_n$. We denote the natural inclusion $\iota_n : \mathbf{H}_n \to \mathbf{H}_{n+1}$. By work of Ocneanu and Jones [Jon87], we have

Proposition 1. *The algebras $\mathbf{H}_n$ carry a unique system of traces Tr_n normalized by the conditions*

$$\begin{aligned} &\mathrm{Tr}_0(1) = 1 && \mathrm{Tr}_n(\sigma_{n-1}\iota_{n-1}(a)) = q^{-1/2} \cdot \mathrm{Tr}_{n-1}(a) \\ &\mathrm{Tr}_n(\iota_{n-1}(a)) = \frac{1-q^{1/2}t^{-1}}{1-q} \cdot \mathrm{Tr}_{n-1}(a) && \mathrm{Tr}_n(\sigma_{n-1}^{-1}\iota_{n-1}(a)) = t^{-1} \cdot \mathrm{Tr}_{n-1}(a). \end{aligned}$$

It is this trace and its generalizations which interest us.

The generalization of this theorem is due to Geck and Lambropoulou [GL97], but one must give more information to obtain a uniqueness statement.

Let $T_{n-1} \in \mathbf{H}(B_n)$ and $U_{2n-1} \in \mathbf{H}(D_{2n})$ be as in paper of Geck and Lambropoulou [GL97]. In all these cases, this is a lift of the longest element of the Weyl group. Under the realizations of these Weyl groups as signed permutations, these longest elements correspond to multiplication by -1.

Proposition 2 ([Gec98]). *For each scalar $y \in \mathbb{C}(q,t)$, there is a unique system of traces Tr_i^y on the algebras $\mathbf{H}(X_n)$ where $X_n = B_n, C_n$ or D_n such that*

$$\begin{aligned} \mathrm{Tr}_0(1) &= 1 && \mathrm{Tr}_n^y(\sigma_{n-1}\iota_{n-1}(a)) = q^{-1/2} \cdot \mathrm{Tr}_{n-1}^y(a) \\ \mathrm{Tr}_n^y(\iota_{n-1}(a)) &= \frac{1-q^{1/2}t}{1-q} \cdot \mathrm{Tr}_{n-1}^y(a) && \mathrm{Tr}_n^y(\sigma_{n-1}^{-1}\iota_{n-1}(a)) = t^{-1} \cdot \mathrm{Tr}_{n-1}^y(a) \\ \mathrm{Tr}_n^y(T_{n-1}\iota_{n-1}(a)) &= y \cdot \mathrm{Tr}_{n-1}^y(a) && \textit{if } X = B, C \\ \mathrm{Tr}_{2n}^y(U_{2n-1}\iota_{n-1}(a)) &= y \cdot \mathrm{Tr}_{n-1}^y(a) && \textit{if } X = D \end{aligned}$$

Remark 1. This is a stronger notion of a Markov trace than used by Geck-Lambropoulou, and stronger than necessary for topological purposes, but our trace will be of this form.

3. **Hecke algebras and geometry.** Let $\mathbb{F}_q$ denote a finite field with q elements. To any Dynkin diagram Γ we may associate a split semi-simple and simply connected group G over $\mathbb{F}_q$. We fix a split Borel subgroup $B \subset G$, and consider the corresponding Bruhat decomposition

$$G(\mathbb{F}_q) = \bigsqcup_{w \in W(\Gamma)} B(\mathbb{F}_q) \cdot w \cdot B(\mathbb{F}_q).$$

A classical theorem of Iwahori identifies the convolution algebra of $B(\mathbb{F}_q) \times B(\mathbb{F}_q)$-invariant $\mathbb{C}$-valued functions on $G(\mathbb{F}_q)$ with $\mathbf{H}(\Gamma)$ specialized at $q \in \mathbb{C}$. Under this identification the element $\sigma_i \in H(\Gamma)$ corresponds to the function which takes the value $q^{-1/2}$ on the orbit $B(\mathbb{F}_q) \cdot s_i \cdot B(\mathbb{F}_q)$.

The fact that the Hecke algebra arises as a convolution algebra suggests a natural categorification using Grothendieck's function-sheaf dictionary. Fix a prime ℓ different from the characteristic of $\mathbb{F}_q$ and let $\Bbbk$ denote an algebraic closure of the field of ℓ-adic numbers.

We denote by $D^b_{B \times B}(G)$ the bounded $B \times B$-equivariant derived category of mixed constructible $\Bbbk$-sheaves on G, often referred to as the **Hecke category**. There is a convolution product on $D^b_{B\times B}(G;\mathbb{k})$ which we denote by $\star$. We fix a square root $q^{1/2}$ of q in $\mathbb{k}$, which allows us to define a square root $(1/2)$ of the Tate twist (1). We denote by $\langle m\rangle$ the shift-twist Functor $[m](m/2)$ (note that $\langle m\rangle$ preserves weight).

Let $\mathbf{IC}_w$ denote the intersection cohomology complex corresponding to the $B\times B$ orbit BwB, normalized so that the restriction of $\mathbf{IC}_w$ to BwB is $\underline{\mathbb{k}}_{BwB}\langle\ell(w)/2\rangle$; that is, we normalize them so that the induced sheaf on G/B is perverse and pure of weight 0.

Finally, let $D_{B\times B}(G)_0$ denote the subcategory of $D_{B\times B}(G)$ consisting of objects isomorphic to direct sums of $\mathbf{IC}_w\langle m\rangle$ for $w\in W(\Gamma)$ and $m\in\mathbb{Z}$ and $\mathscr{K}_{B\times B}(G)_0$ its split Grothendieck group. Then $D_{B\times B}(G)_0$ is preserved under convolution, and $\star$ induces a ring structure on $\mathscr{K}_{B\times B}(G)_0$. The assignment $q^{1/2}\mapsto\langle -1\rangle$ makes $\mathscr{K}_{B\times B}(G)$ into an algebra over $\mathbb{Z}[q^{1/2},q^{-1/2}]$. The following result is well-known:

Theorem 3 (See [Spr82]). *The Hecke algebra $\mathbf{H}(\Gamma)$ can be identified with $\mathscr{K}_{B\times B}(G)_0$. Under this identification the Kazhdan-Lusztig basis C'_w corresponds to the class of the intersection cohomology complex $[\mathbf{IC}_w]$.*

The relationship between these sheaves and the Hecke algebra was an important ingredient in the Kazhdan-Lusztig conjecture [KL79, KL80].

Alternatively, one can consider the $B\times B$-equivariant cohomology of $\mathcal{F}_a$. This a **Soergel bimodule** $\mathbb{H}^*_{B\times B}(G;\mathcal{F}_a)=S_a$ (for more abckground on Soergel bimodules and their connections to geomtery, see our earlier paper [WW08] or the original paper [Soe92] of Soergel).

4. **A categorified trace.** Let $B_\triangle\subset B\times B$ be the diagonal subgroup. Then we can associate to $\mathcal{F}_a$ and S_a bigraded vector spaces $\mathbb{H}^*_{B_\triangle}(\mathcal{F}_a)$ and $HH^*(S_a)$.

The bigrading on $HH^*(S_a)$ is easy to describe: one grading, which we call the t**-grading**, is given by the homological grading on HH^* and one is given by the fact that S_a is a graded bimodule over polynomials, with linear polynomials in degree 2. For reasons of knot theory, we follow the convention of [MSV] with respect to the grading on Hochschild homology: we normalize the gradings so that the differential on Hochschild homology has bidegree $(1,1)$.

The bigrading on $\mathbb{H}^*_{B_\triangle}(\mathcal{F}_a)$ is of geometric origin, but less obvious than the algebraic grading. What we must use is the weight grading on cohomology. By previous work of the authors [WW08, Theorem 1.4], there is an isomorphism $\mathbb{H}^*_{B_\triangle}(\mathcal{F}_a)\cong HH^*(S_a)$, which sends the q-grading to the usual cohomological grading and the t-grading to the weight plus cohomological grading, i.e. that given by the norm of the eigenvalue of Frobenius.

Thus, we let $\mathbb{H}^{i;j}_{B_\triangle}(G;\mathcal{F})$ be subspace of $\mathbb{H}^{i;j}_{B_\triangle}(G;\mathcal{F})$ of weight j. In all applications encountered below, Froenius will always act via powers of our fixed $q^{1/2}$ and hence $\mathbb{H}^{i;j}_{BD}(G,\mathcal{F})$ is simply the $q^{j/2}$-eigenspace for the action of Frobenius.

Definition 4. *The* **mixed Poincaré polynomial** *of this sheaf is the sum*

$$P_{\mathcal{F}_a}(q,t)=\dim_{q,t}\mathbb{H}^*_{B_\triangle}(G;\mathcal{F}_a)=\sum_{i,j}\dim\mathbb{H}^{i;j}_{B_\triangle}(G;\mathcal{F}_a)q^{i/2}t^{j-i}.$$

We will always consider equivariant mixed Poincaré polynomials, and only specify the group if there is danger of confusion. Note that

$$P_{\mathcal{F}_a\langle -i\rangle}(q,t)=q^{i/2}P_{\mathcal{F}_a}(q,t). \tag{1}$$

As before, Γ is a Dynkin diagram, and we let $\Gamma' = \Gamma - \{v\}$ for a single vertex v, and $\iota : \mathbf{H}(\Gamma') \to \mathbf{H}(\Gamma)$ is the induced inclusion.

Theorem 5. *The linear extension of the function* $\mathrm{Tr}_\Gamma(a) = \dim_{q,t} \mathbb{H}^*_{B_\Delta}(\mathcal{F}_a)$ *is a trace on* $\mathbf{H}(\Gamma)$*. Furthermore, for each vertex* v*, we have*

$$\begin{aligned} &\mathrm{Tr}_\emptyset(1) = 1 && \mathrm{Tr}_\Gamma(\sigma_v \iota(a)) = -q^{-1/2} t \cdot \mathrm{Tr}_{\Gamma'}(a) \\ &\mathrm{Tr}_\Gamma(\iota(a)) = \frac{1+qt}{1-q} \cdot \mathrm{Tr}_{\Gamma'}(a) && \mathrm{Tr}_\Gamma(\sigma_v^{-1} \iota(a)) = q^{-1/2} \cdot \mathrm{Tr}_{\Gamma'}(a) \end{aligned}$$

In particular, when $\Gamma = A_n$*, this trace coincides with that of Jones-Ocneanu. Furthermore, in the case where* $\Gamma = B_n, C_n, D_n$*, it is the trace* $\mathrm{Tr}_\Gamma^{q^{-1/2}}$ *given in Proposition 2, where* $y = q^{-1/2}$*.*

This establishes that this trace pulled back to the braid group (up to normalization) provides a knot invariant, called the HOMFLYPT polynomial in the type A case, and an invariant of knots in the solid torus in type B, by the "cylindrical Markov theorem."

Proof of Theorem 5. This is, in fact, a trace since

$$\mathbb{H}^*_{B_\Delta}(G; \mathcal{F} \star \mathcal{F}') \cong \mathbb{H}^*_{B_\Delta \times B_\Delta}(G \times G; \mathcal{F} \boxtimes \mathcal{F}') \cong \mathbb{H}^*_{B_\Delta}(G; \mathcal{F}' \star \mathcal{F}).$$

Thus, we need only establish the normalization conditions. We establish both of these through a slightly stronger condition.

We can choose a Borus $T \subset B \subset G$ for G such that $T' = G' \cap T \subset B' = G' \cap B \subset G'$ is a Borus for G'. Let $P \subset G$ be the minimal parabolic containing a representative for s_v. Consider the closed embeddings

$$\mathfrak{I} \colon B \times_{B'} G' \to G \qquad \mathfrak{S} \colon P \times_{B'} G' \to G,$$

Both these maps are embeddings with $T \times T$ invariant image and thus induce actions on the domains. These are geometric versions of $\iota(-)$ and $C'_v \cdot \iota(-)$ (where as before $C'_v = (\sigma_v + q^{1/2}) = [\underline{\Bbbk}_P\langle 1\rangle]$) in the sense that

$$\mathfrak{I}_*(\underline{\Bbbk}_B \star \mathcal{F}_a) = \mathcal{F}_{\iota(a)} \qquad \mathfrak{S}_*(\underline{\Bbbk}_P \star \mathcal{F}_a\langle 1\rangle) = \mathcal{F}_{C'_v \iota(a)}.$$

Let $Z = Z_T(G')$, let Y be the semisimple part of the Levi of P, and w'_0 is a representative for the longest element of the Weyl group for G'. By standard Lie theory, we have decompositions

$$B = Z(N \cap w'_0 N w'_0) B' \qquad P = Y(N \cap w'_0 N w'_0) B'$$

And thus we have $T_\Delta = Z_\Delta \times T'_\Delta$-equivariant deformation retracts,

$$B \times_{B'} G' \simeq Z \times G' \qquad P \times_{B'} G' \simeq Y \times G'$$

where the right hand side of both equations carries the exterior product action for Z_Δ on P and T'_Δ on G', since $[Z, B'] = [Y, T'] = \{1\}$.

It follows that we have isomorphisms of graded vector spaces compatible with mixed structures

$$\begin{gathered} \mathbb{H}^*_{B_\Delta}(G; \mathcal{F}_{\iota(a)}) \cong \mathbb{H}^*_{\mathbb{G}_m}(\mathbb{G}_m) \otimes_\Bbbk \mathbb{H}^*_{B'_\Delta}(G'; \mathcal{F}_a), \\ \mathbb{H}^*_{B_\Delta}(G; \mathcal{F}_{C'_v \iota(a)}) \cong \mathbb{H}^*_{\mathbb{G}_m}(SL_2) \otimes_\Bbbk \mathbb{H}^*_{B'_\Delta}(G'; \mathcal{F}_a) \end{gathered}$$

Applying $\dim_{q,t}$, we get the relations

$$\mathrm{Tr}(\iota(a)) = \frac{1+qt}{1-q}\mathrm{Tr}(a)$$
$$\mathrm{Tr}(C'_v\iota(a)) = \frac{q^{-1/2}+q^{3/2}t}{1-q}\mathrm{Tr}(a)$$

This establishes one of our conditions. For the others, we first observe that $\sigma_v = C'_v - q^{-1/2}$ and $\sigma_v^{-1} = C'_v - q^{1/2}$. By basic algebra, we find that

$$\begin{aligned}\mathrm{Tr}(\sigma_v\iota(a)) &= \frac{q^{3/2}t - q^{1/2t}}{1-q}\cdot\mathrm{Tr}(a) \\ &= -q^{1/2}t\cdot\mathrm{Tr}(a)\end{aligned} \qquad \begin{aligned}\mathrm{Tr}(\sigma_v^{-1}\iota(a)) &= \frac{q^{-1/2}-q^{1/2}}{1-q}\cdot\mathrm{Tr}(a) \\ &= q^{-1/2}\cdot\mathrm{Tr}(a).\end{aligned}$$

Finally, we must establish our claim that

$$\mathrm{Tr}_{B_{n+1}}(T_n\iota(a)) = q^{-1/2}\cdot\mathrm{Tr}_{B_n}(a)$$

This follows from Corollary 13, which shows that our trace coincides with that considered by Gomi and from [Gom06, §4.4 & 4.5] where Gomi shows that his trace corresponds to that constructed by Geck and Lambropoulou, which has this property by [GL97, Proposition 4.5]. □

5. **Character sheaves and character expansions.** Of course, any trace on a semi-simple algebra can be canonically expanded as a direct sum of the traces on irreducible representations, i.e. as a sum of irreducible characters. A formula for this was given by in type A by Ocneanu [Wen88], by type B by Orellana [Ore99], and in arbitrary type by Gomi [Gom06]. Using our geometric perspective, we can give a much simpler proof than that of [Gom06], which depended on case by case analysis.

Our proof is founded by the fact that the expansion of the Hecke algebra as a sum of matrix algebras has a geometric manifestation: given a simple perverse sheaf $\mathbf{IC}_w$, we can apply the "averaging" functor $\mathrm{ind}_{B_\Delta}^{G_\Delta}\colon D^b_{B_\Delta}(G)\to D^b_{G_\Delta}(G)$. Since G/B is projective, by the Decomposition Theorem, each summand of $\mathbf{K}_w = \mathrm{ind}_{B_\Delta}^{G_\Delta}\mathbf{IC}_w$ is a shift of a simple perverse sheaf.

Definition 6. *A simple perverse sheaf in* $D^b_{G_\Delta}(G)$ *is called a* **unipotent character sheaf** *if it appears as a summand of* $\mathbf{K}_w$ *for some* w. *Following* [MS89, §5.2], *we let* $\hat{G}(1)$ *denote the set of unipotent character sheaves.*

We choose our normalization so that each $\mathcal{V}\in\hat{G}(1)$ extends a local system of weight 0 in degree 0. In particular, each $\mathcal{V}\in\hat{G}(1)$ is pure of weight zero. For each character $\chi\in\hat{W}$ there exists a unipotent character sheaf $\mathcal{V}_\chi\in G(1)$ whose restriction to the regular semisimple elements of G is a local system with monodromy given by χ. Furthermore, we denote by $\widehat{G(1)}_{ex}$ those character sheaves in $\hat{G}(1)$ which are not of the form A_χ for $\chi\in\hat{W}$.

Since induction is adjoint to restriction, we have a natural isomorphism

$$\mathbb{H}^*_{G_\Delta}(G;\mathbf{K}_w)\cong\mathbb{H}^*_{B_\Delta}(G;\mathbf{IC}_w).$$

The decomposition of $\mathbf{K}_w$ into simple character sheaves mirrors that of C'_w into operators on the various irreducible representations, so we can write the weights of our trace in terms of the mixed Poincaré polynomials of simple character sheaves. Let us make this precise.

For $\mathcal{V} \in \hat{G}(1)$, we let $P_{\mathcal{V}}(q,t)$ denote the G_Δ-equivariant mixed Poincaré polynomial of $\mathcal{V}$. Let $\{\chi', \chi\}$ be the matrix coefficents of Lusztig's "Fourier transform."

Proposition 7. *The weight ϖ_χ of the character χ in* Tr *is*

$$\varpi_\chi = \sum_{\chi' \in \hat{W}} \{\chi', \chi\} \cdot P_{\mathcal{V}_{\chi'}}(q,t)$$

Proof. For the course of the proof consider $\mathscr{S}$ the split Grothendieck group of $D^b_{G_\Delta}(G)$. If $\mathcal{F} \in D^b_{G_\Delta}(G)$ we denote by $[\mathcal{F}]$ its class in $\mathscr{S}$ and, given a polynomial $a = a_i q^{i/2} \in \mathbb{Z}[q^{\pm 1/2}]$ we set

$$a \cdot [\mathcal{F}] := \sum_i a_i [\mathcal{F}\langle -i \rangle] \in \mathscr{S}.$$

In this notation, combining Lusztig's formula [Lus85c, 14.11] and [Lus86, 23.1] we obtain

$$[\mathbf{K}_w] = \sum_{\chi, \chi' \in \hat{W}} \chi_q(C'_w)\{\chi', \chi\} \cdot [\mathcal{V}_{\chi'}] + \sum_{\mathcal{V} \in \widehat{G(1)}_{ex}} a_{\mathcal{V}} \cdot [\mathcal{V}]. \tag{2}$$

(Some remarks are in order. Firstly, note that for Lusztig's objects $\bar{K}_w$ we have $\bar{K}_w\langle \ell(w) \rangle \cong \mathbf{K}_w$ and Lusztig normalizes character sheaves so as to be perverse. Secondly, note that [Lus85c, 14.11] is an equality after forgetting mixed structures, however this can be lifted to the mixed setting by using the fact that $\mathbf{K}_w$ is pure of weight zero. Thirdly, the sign $\epsilon_{\mathcal{V}_\chi}$ (see [Lus85c, 13.10]) is equal to $(-1)^{\dim G}$ as A_χ occurs as a direct summand of the Springer sheaf $\mathbf{K}_{id}$.)

We will see below that the G_Δ-equivariant cohomology of any $\mathcal{V} \in \hat{G}(1)_{ex}$ is zero. Hence, applying hypercohomology to (2) we obtain (note also (1))

$$\mathrm{Tr}(C'_w) = P_{\mathbf{IC}_w}(q,t) = P_{\mathbf{K}_w}(q,t) = \sum_{\chi \in \hat{W}} \chi_q(C'_w) \Big(\sum_{\chi' \in \hat{W}} \{\chi', \chi\} \cdot P_{\mathcal{V}_{\chi'}}(q,t) \Big)$$

and the proposition follows. □

Thus, the primary point remaining to us is the computation of the mixed Poincaré polynomial $P_{\mathcal{V}}(q,t)$. Our first step is to show that a number of character sheaves have trivial cohomology.

If $L \subset G$ is the Levi of a parabolic, then there are adjoint **parabolic restriction** and **parabolic induction** functors

$$\begin{tikzcd} D^b_{L_\Delta}(L) \arrow[r, bend right, "\mathrm{Ind}_L^G"'] & D^b_{G_\Delta}(G) \arrow[l, bend right, "\mathrm{Res}_L^G"'] \end{tikzcd}$$

relating the conjugation-equivariant derived categories of these groups, defined by Lusztig [Lus85a]. These functors are different from the usual restriction and induction functors on equivariant derived categories, which we write uncapitalized.

For each character sheaf $\mathcal{V}$, there is up to conjugacy a unique minimal Levi $L_{\mathcal{V}}$ such that $\mathcal{V}$ is a summand of $\mathrm{Ind}_L^G \mathcal{V}'$ where $\mathcal{V}'$ is a strongly cuspidal character sheaf on L. This is also the minimal Levi for which $\mathrm{Res}_L^G \mathcal{V} \neq 0$.

Even in the equivariant derived category $D_{G_\Delta}(G)$, these Levis index a block decomposition of the category generated by character sheaves. That is,

Proposition 8. *If $\mathcal{V}, \mathcal{W}$ are character sheaves and $\operatorname{Ext}^\bullet_{G_\Delta}(\mathcal{V}, \mathcal{W}) \neq \{0\}$, then $L_\mathcal{V}$ is conjugate to $L_\mathcal{W}$.*

In particular, if $L_\mathcal{V} \neq T$, then $\mathbb{H}^*_{G_\Delta}(G; \mathcal{V}) = \{0\}$ and $P_\mathcal{V}(q,t) = 0$.

Proof. This was proved in the non-equivariant situation by Lusztig in [Lus85b, 7.2], and the equivariant result follows by the Leray-Serre spectral sequence. ☐

One simple corollary of Proposition 8 is that:

Theorem 9. *Any unipotent character sheaf is G_Δ-equivariantly formal.*

Proof. If $\mathcal{V}$ is induced from a cuspidal sheaf on a non-abelian Levi, then by Proposition 8, $\mathbb{H}^*_{G_\Delta}(\mathcal{V}) = 0$, so $\mathcal{V}$ is equivariantly formal.

On the other hand, every character sheaf not covered by the previous assertion is a summand of $\mathbf{K}_1$, which is equivariantly formal, since $\mathbf{IC}_1 = \Bbbk_B$ is. ☐

This allows us to answer definitively a question answered positively for type A in a previous paper of the authors [WW08].

Corollary 10. *For any group G and any $w \in W$, the sheaf $\mathbf{IC}_w$ is B_Δ-equivariantly formal.*

Proof. The sheaf $\mathbf{K}_w$ is equivariantly formal, since it is a sum of unipotent character sheaves.

By [GKM98], a sheaf is G-equivariantly formal if and only if its cohomology is torsion-free for the action of $R_G = H^*(BG)$, and similarly for B. Thus,

$$\operatorname{Tor}_{R_G}(\mathbb{H}^*_{G_\Delta}(\mathbf{K}_w)) \cong \operatorname{Tor}_{R_G}(\mathbb{H}^*_{B_\Delta}(\mathbf{IC}_w)) = \{0\}.$$

Assume $x \in \operatorname{Tor}_{R_B}(\mathbb{H}^*_{B_\Delta}(\mathbf{IC}_w))$. Then by definition, there is some polynomial $p \in R_B$, such that $px = 0$. On the other hand, this means that $\left(\prod_{w\in W} p^w\right) x = 0$, and so we have $x \in \operatorname{Tor}_{R_G}(\mathbb{H}^*_{B_\Delta}(\mathbf{IC}_w)) = \{0\}$. This establishes that $\operatorname{Tor}_{R_B}(\mathbb{H}^*_{B_\Delta}(\mathbf{IC}_w)) = \{0\}$ and thus the corollary. ☐

6. **Character sheaves and Molien series.** Proposition 8 shows that in equation (7), we need only sum over the sheaves that appear as summands of $\mathbf{K}_1$, since the trivial local system is the only unipotent character sheaf on T. Let us give a more careful description of these sheaves.

Consider the variety $\tilde{G} = G_\Delta \times_{B_\Delta} B$ and let $c : G_\Delta \times_{B_\Delta} B \to G$ be the map induced by $(g,t) \mapsto gtg^{-1}$. A unipotent character sheaf is induced from T if it is a summand of $\mathbf{K}_1 = c_*\underline{\Bbbk}_{\tilde{G}}$.

A theorem of Ginzburg [Gin93, Theorem 8.1] shows that the category of mixed perverse sheaves generated by $\mathbf{K}_1 = c_*\underline{\Bbbk}_{\tilde{G}}$ is equivalent to the category of graded modules over the smash product algebra $\Bbbk[W] \# \operatorname{Sym}^\bullet(\mathfrak{t})$, or in its Koszul dual description

$$\operatorname{Ext}^\bullet_{\mathsf{Perv}}(\mathbf{K}_1) \cong H^{BM}_*(\tilde{G} \times_G \tilde{G}) \cong \Bbbk[W] \# (\wedge^\bullet \mathfrak{t}^*)$$

In particular, there are canonical bijections between the set $\hat{W}$ of irreducible W-representations V_χ over $\Bbbk$ and the set of unipotent character sheaves $\mathcal{V}_\chi$ induced from T.

We will require a version of this theorem which incorporates the richer structure of the equivariant derived category. Consider the algebra $E = H^*_{T_\Delta}(T) \cong \operatorname{Sym}^\bullet(\mathfrak{t}^*) \otimes \wedge^\bullet \mathfrak{t}^*$ endowed with the tensor product action of W. The geometric realization of this algebra gives it q- and t- gradings.

Proposition 11. *We have a natural isomorphism* $\operatorname{Ext}^\bullet_{G_\Delta}(\mathbf{K}_1) \cong \Bbbk[W]\#E$ *as bigraded algebras. In particular, since* $\underline{\Bbbk}_G \cong \mathcal{V}_{\mathbb{1}}$*,*

$$\operatorname{Ext}^\bullet_{G_\Delta}(\mathcal{V}_\chi, \mathcal{V}_\nu) = \operatorname{Hom}_{W\times W}(V_\chi^* \boxtimes V_\nu, \Bbbk[W]\#E) \qquad \operatorname{Ext}^\bullet_{G_\Delta}(\underline{\Bbbk}_G, \mathcal{V}_\nu) = \operatorname{Hom}_W(V_\nu, E).$$

Thus the bigraded dimension of the cohomology of these character sheaves is given by a frequently studied combinatorial invariant.

Definition 12. *The graded dimension*

$$M_\nu(q,t) = \sum_{q,t} q^i t^j \dim \operatorname{Hom}_W(V_\nu, E)^{i,j}$$

is called the **Molien series** *of* ν*.*

This geometric realization of the Molien series allows us to arrive at the main theorem of [Gom06] by a very different route.

Corollary 13. *For all characters* $\nu \in \hat{W}$*, we have* $M_\nu(q,t) = P_{\mathcal{V}_\nu}(q,t)$*. In particular, by Propositon 7, it follows that*

$$\operatorname{Tr}(x) = \sum_{\chi\in\hat{W}} \chi_q(x) \cdot \{\chi, \nu\} \cdot M_\nu(q,t).$$

We thank Victor Ginzburg for helpful suggestions on the composition of this proof.

Proof of Proposition 11. First, we note that this is correct on the level of vector spaces. We have an isomorphism of vector spaces (not grading preserving!)

$$\operatorname{Ext}^\bullet_{G_\Delta}(\mathbf{K}_1) \cong H_*^{BM,G_\Delta}(\tilde{G} \times_G \tilde{G}; \Bbbk)$$

By a theorem of Ginzburg [CG97, Theorem 8.6.7], this is an isomorphism of algebras when the RHS is given the convolution product.

We have a natural map $\tilde{G} \to T$ induced from the B_Δ-invariant map $B \to T$. Let $U \subset \tilde{G}$ be the preimage of $1 \in T$ (that is, the set where the corresponding group element is unipotent).

There is an equivariant map $\tilde{G} \times_G \tilde{G} \to T$ given by applying the standard "ordered eigenvalues" map on the first factor. This map is a topologically trivial bundle with fiber given by the Steinberg variety $U \times_G U$, so the isomorphism $H_*^{BM,G_\Delta}(\tilde{G}\times_G\tilde{G}; \Bbbk) \cong \Bbbk[W]\#E$ follows immediately, since by [CG97, Theorem 7.2.2], we have an algebra isomorphism $H_*^{BM,G_\Delta}(U \times_G U; \Bbbk) \cong \Bbbk[W]\# \operatorname{Sym}^\bullet(\mathfrak{t})$.

This argument further shows that the pullback map is surjective and an isomorphism on the pure subalgebra of $\operatorname{Ext}^\bullet_{G_\Delta}(\mathbf{K}_1)$.

$$\operatorname{Ext}^\bullet_{G_\Delta}(\mathbf{K}_1) \to \operatorname{Ext}^\bullet_{G_\Delta}(c_*\underline{\Bbbk}_U) \cong \Bbbk[W]\# \operatorname{Sym}^\bullet(\mathfrak{t}),$$

On the other hand, we also have a surjective map given by forgetting equivariance

$$\operatorname{Ext}^\bullet_{G_\Delta}(\mathbf{K}_1) \to \operatorname{Ext}^\bullet(\mathbf{K}_1) \cong \Bbbk[W]\#(\wedge^\bullet \mathfrak{t}^*)$$

This map is an isomorphism on the subalgebra where the q and t gradings coincide.

Thus, we know that our algebra is generated by subalgebras isomorphic to $\Bbbk[W]\# \operatorname{Sym}^\bullet(\mathfrak{t})$ and $\Bbbk[W]\#(\wedge^\bullet \mathfrak{t}^*)$, so we need only show that the included copies of $\operatorname{Sym}^\bullet(\mathfrak{t})$ and $(\wedge^\bullet \mathfrak{t}^*)$ commute. These are given by the image of $\operatorname{Ext}^\bullet_{G_\Delta}(\underline{\Bbbk}_{\tilde{G}}, \underline{\Bbbk}_{\tilde{G}}) \cong E$ under the obvious inclusion (one can check this by composing this inclusion with the projections described above). This ring is commutative since it is the equivariant cohomology of a space. □

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