2-Block Springer Fibers and Disoriented Knot Homology

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Nilpotents in Mat(n, n)

Definition

We call an element $N \in \mathrm{Mat}(n, n)$ **nilpotent** *if* $N^n = 0$.

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Definition

We define the Springer fiber of N to be the set of flags preserved by N.

$$X_N = \{\underline{F} \in X | N(\underline{F}) \subset \underline{F}\}$$

Young diagrams

For any nilpotent, the sizes of the Jordan blocks form a partition, which we can represent as a Young diagram. The number of rows is the number of Jordan blocks.

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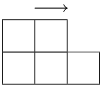

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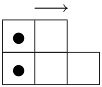

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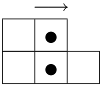

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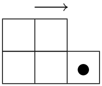

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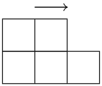

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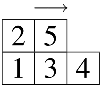

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By a general result of Spaltenstein, the components $\mathcal{X}_S$ of X_N are in bijection with standard Young tableaux of the same shape.

A **standard tableau** on a diagram is a filling of each box with the numbers $[1, n]$, such that both columns and rows are strictly increasing.

Cup diagrams

For now on, we'll restrict to 2 row diagrams of shape (k, k).

Proposition

There is a bijection from standard Young tableaux of shape (k, k), to noncrossing cup diagrams on $2k$ points (right ends of cups=top row).

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Let σ be the involution given by the cups, and let $\delta(i)$ is the number of cups "nested inside" the one from i to $\sigma(i)$. That is, $\delta(i) = (|\sigma(i) - i| - 1)/2$.

Components of X_N

Proposition (Fung, 2003)

The flag $\mathcal{F}$ lies in the component $\mathcal{X}_S$ if and only if for all i in the bottom row,

$$N^{\delta(i)+1}(F_{\sigma_S(i)}) = F_{i-1}.$$

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For example, when $n = 4, k = 2$:

$$\mathcal{X}_{\begin{smallmatrix}3&4\\1&2\end{smallmatrix}} = \{F_1 \subset F_2 \subset F_3 \subset \mathbb{C}^4 | N(F_3) = F_1\}$$

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Proposition (Fung)

$\mathcal{X}_S$ is an iterated $\mathbb{P}^1$ bundle of dimension k (topologically trivial, not holomorphically). In particular, $\dim H^*(\mathcal{X}_S) = 2^k$.

Intersections of components

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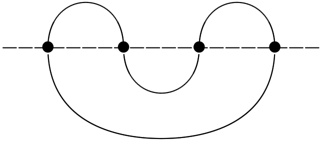

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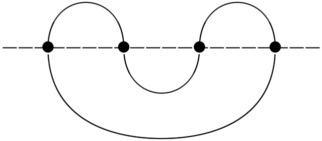

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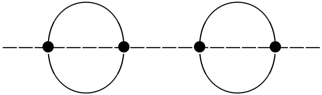

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We have maps $R \to H^*(X) \to H^*(X_N)$ given by taking Chern classes $(x_i \mapsto c_1(V_i/V_{i-1}))$.

Theorem

We have isomorphisms

$$H^*(\mathcal{X}_S) \cong R/(x_i + x_{\sigma(i)}, x_i^2) \cong \left(\mathbb{C}[x]/(x^2)\right)^{\otimes k}$$

$$H^*(\mathcal{X}_S \cap \mathcal{X}_{S'}) \cong R/(x_i + x_{\sigma_S(i)}, x_i + x_{\sigma_{S'}(i)}, x_i^2) \cong \left(\mathbb{C}[x]/(x^2)\right)^{\otimes k_{S,S'}}$$

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Since pullback is functorial, this tells us about the bimodule multiplication

$$H^*(\mathcal{X}_S) \otimes H^*(\mathcal{X}_S \cap \mathcal{X}_{S'}) \otimes H^*(\mathcal{X}_{S'}) \to H^*(\mathcal{X}_S \cap \mathcal{X}_{S'}).$$

Combinatorial multiplication

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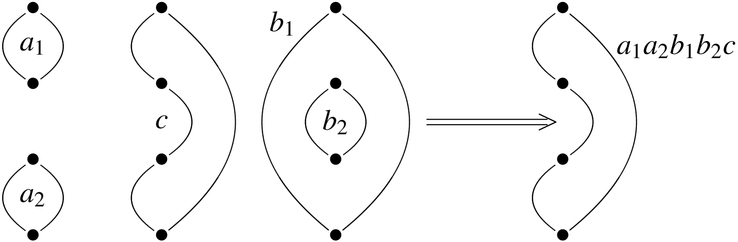

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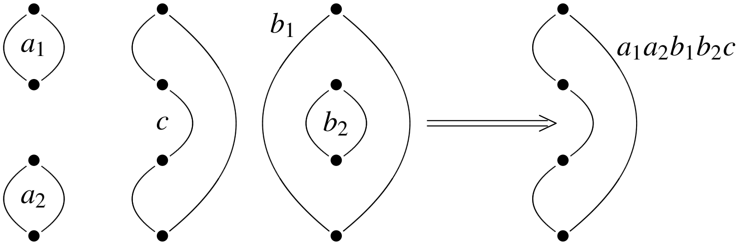

Maybe some of you have seen this sort of multiplication before: it appears in Khovanov's algebra $\mathcal{H}^k$.

Convolution

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Theorem (Stroppel-W.)

Our combinatorial picture with circles gives an isomorphism of vector spaces

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This doesn't seem to be an algebra isomorphism! But it's very close!

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Proposition

As algebras, we have an isomorphism,

$$\mathrm{Ext}_{\mathsf{Fuk}(Y_N)}\left(\bigoplus_S \mathcal{X}_S, \bigoplus_S \mathcal{X}_S\right) \cong \mathcal{S}^k.$$

A historical interlude

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- Any TQFT applied to H^k gives an honest algebra. Khovanov chooses the one attached to $H^*(\mathbb{P}^1)$ with the usual Frobenius structure. That is, we associate a copy of $H^*(\mathbb{P}^1)$ to each circle, and take the tensor product of the algebras when we take union of the circles (and take direct sum for formal sum of manifolds).

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Khovanov then uses this algebra to define a knot invariant.

A little knot theory

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The unoriented picture

$$\begin{array}{ccc} \mathcal{H}^k-\text{modules} & \longleftarrow & \text{knots} \\ \downarrow & & \downarrow{\scriptstyle BN} \\ \text{Khovanov homology} & \longleftarrow & \text{complexes of cobordisms} \end{array}$$

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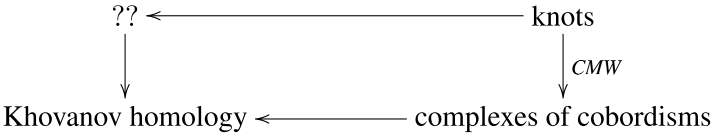
Then Clark, Morrison and Walker modified Bar-Natan's picture to use a knot invariant valued in a **disoriented** cobordism category, which fixed the sign problem in Khovanov homology.

The disoriented picture



Our algebra vs. Khovanov's

Recall that the CMW theory has base ring $S = \mathbb{C}[\omega]/(\omega^4)$, that is, it has a parameter ω which can be specialized at $\omega = \pm 1$ or $\omega = \pm i$.

Proposition (S.-W.)

There's an algebra object in CMW's disoriented cobordism category such that application of Bar-Natan's TQFT gives an algebra A over S satisfying

- *At* $\omega = \pm 1$, *the algebra A specializes to* $\mathcal{H}^k$.
- *At* $\omega = \pm i$, *the algebra A specializes to* $\mathcal{S}^k$.

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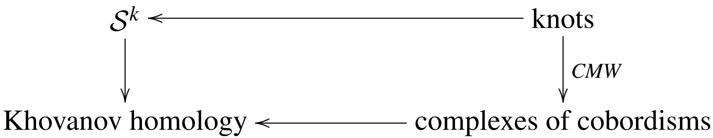

Generalizations *(joint w/ Braden, Proudfoot and Licata)*

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There is a combinatorial duality on hyperplane arrangements $\mathcal{V} \leftrightarrow \mathcal{V}^\vee$.

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This relates, for example, the Fukaya categories of $T^*\mathbb{P}^n$ and $\widetilde{\mathbb{C}^2/\mathbb{Z}_n}$. This seems to be a special case of one manifestation of 3-d mirror symmetry.

Thanks, y'all.