Geometry of Soergel Bimodules

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IAS/MIT

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References:

This slide show can be downloaded from

http://math.berkeley.edu/˜bwebste/faro-slides.pdf

Some references:

- B. W. and G. W., *A geometric model for the Hochschild homology of Soergel bimodules*. (**http://math.berkeley.edu/˜bwebste/hochschild-soergel.pdf**)
- W. Soergel, *The combinatorics of Harish-Chandra bimodules*.
- W. Soergel, *Kategorie O, Perverse Garben und Moduln über den Koinvarianten zur Weylgruppe*.
- M. Khovanov, *Triply-graded link homology and Hochschild homology of Soergel bimodules*.
- J. Bernstein and V. Lunts, *Equivariant sheaves and functors*.

Soergel bimodules

Let $R = \mathbb{C}[x_1, \ldots, x_n]/(x_1 + \cdots + x_n)$, and s_i be the map permuting x_i and x_{i+1} and let $G = SL(n, \mathbb{C})$, and $B \subset G$ upper-triangular matrices.

Like so many objects in mathematics, Soergel bimodules have a number of definitions:

1. One which explains why anyone ever cared:

Definition

A **Soergel bimodule** *is the image of a projective object in category* $\tilde{O}$ *under Soergel's "combinatoric" functor* $\mathbb{V}$.

2. One which is hands-on but totally unilluminating:
3. One which involves disgusting levels of machinery, but which ultimately is the best for working with:

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Definition

A **Soergel bimodule** *is a direct sum of summands of tensor products*

$$R_{\mathbf{i}} \cong R \otimes_{R^{s_{i_1}}} R \otimes_{R^{s_{i_2}}} \cdots \otimes_{R^{s_{i_m}}} R$$

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Definition

A **Soergel bimodule** *is the hypercohomology of a semi-simple $B \times B$-equivariant perverse sheaf on G.*

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3. One which involves disgusting levels of machinery, but which ultimately is the best for working with: *perverse sheaves*

While intimidating at first, a multiplicity of definitions is, in fact, a strength rather than a weakness, allowing us to our problems translate back and forth at will.

Soergel bimodules for $n = 2$

When $n = 2$, then

$$R = \mathbb{C}[x_1, x_2]/(x_1 + x_2) \cong \mathbb{C}[y]$$

with the action of s sending $y \mapsto -y$. Thus, $R^s = \mathbb{C}[y^2]$ and

$$R_1 \cong R \otimes_{R^s} R \cong \mathbb{C}[y \otimes 1, 1 \otimes y] \cdot r/(y^2 \otimes 1 - 1 \otimes y^2)$$

Proposition

The elements $1 \otimes 1 \otimes 1$ and $1 \otimes y \otimes 1$ generate $R \otimes_{R^s} R \otimes_{R^s} R$ as an R-bimodule, and generate two summands, so $R \otimes_{R^s} R \otimes_{R^s} R \cong R_1 \oplus R_1\{2\}$.

Corollary

Every indecomposable Soergel bimodule for $n = 2$ is isomorphic to R or R_1.

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When $n = 3$, similar calculations show

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Every indecomposable Soergel bimodule for $n = 2$ is isomorphic to one of R, $R \otimes_{R^{S_3}} R$ or

$$R_1 \cong R \otimes_{R^{s_1}} R \qquad R_{12} \cong R_1 \otimes R_2 \cong R \otimes_{R^{s_1}} R \otimes_{R^{s_2}} R$$
$$R_2 \cong R \otimes_{R^{s_2}} R \qquad R_{21} \cong R_2 \otimes R_1 \cong R \otimes_{R^{s_2}} R \otimes_{R^{s_1}} R,$$

Anyone used to playing with $\mathrm{SL}(3)$ will probably note that we have an obvious bijection from S_3 to the set of indecomposable Soergel bimodules:

$$1 \leftrightarrow R \qquad (12) \leftrightarrow R_1 \qquad (23) \leftrightarrow R_2$$
$$(123) \leftrightarrow R_{21} \qquad (132) \leftrightarrow R_{12} \qquad (13) \leftrightarrow R \otimes_{R^{S_3}} R$$

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Indecomposable Soergel bimodules

Question

In general, is the set of indecomposable Soergel bimodules in bijection with S_n?

Definition 2 is perfectly useless at answering this sort of question. But from the perspectives of Definitions 1 or 3, it borders on obvious:

Theorem (Soergel)

For each $w \in S_n$, there is a indecomposible Soergel bimodule R_w (and these are pairwise not isomorphic). For any reduced expression $w = s_{i_1} \cdots s_{i_m}$, R_w is the bimodule generated by $1 \otimes \cdots \otimes 1$ in $R_{\mathbf{i}}$.

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The Rouquier complex and triply graded homology

Why are these interesting?

Theorem (Soergel, 1992)

The category of Soergel bimodules is a categorification of the Hecke algebra.

Theorem (Rouquier, 2004)

For each braid σ on n strands, there is a complex of Soergel bimodules $F(\sigma)$ such that

$$F(\sigma) \otimes F(\sigma') \simeq F(\sigma\sigma')$$

categorifying the usual map of the braid group to the Hecke algebra.

Theorem (Khovanov-Rozansky,2006)

The homology of the complex $HH_*(F(\sigma))$ is a triply graded knot homology theory categorifying the HOMFLY polynomial.

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Hochschild homology

Definition

If M is a bimodule over a ring R, and $\mathbf{P}^\bullet$ *is a free resolution of R as a bimodule over itself, then the* **Hochschild homology** $HH_*(M)$ *is the homology of* $M \otimes_{R\otimes R} \mathbf{P}^\bullet$.

If $R = k[V]$ is a polynomial ring, we have a very convenient free resolution, called the **Koszul complex** of the form

$$R \otimes R \longleftarrow R \otimes R \otimes V^* \longleftarrow R \otimes R \otimes \wedge^2 V^* \longleftarrow \cdots \longleftarrow R \otimes R \otimes \wedge^n V^* \longleftarrow 0$$

Since the modules appearing in $F(\sigma)$ are of the form $R_{\mathbf{i}}$ for $\mathbf{i}$'s connected to the factorization of σ into braid generators, understanding $HH_*(R_{\mathbf{i}})$ is a question of interest to anyone who thinks about in triply graded homology.

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The Hochschild homology of $R_{\mathbf{i}}$

Theorem (W.-Williamson)

There exists a compact smooth manifold $X_{\mathbf{i}}$ (a "groupy Bott-Samelson space") such that there is a natural isomorphism of graded R-modules

$$HH_*(R_{\mathbf{i}}) \cong R \otimes_{\mathbb{C}} H^*(X_{\mathbf{i}})$$

where $HH_*(R_{\mathbf{i}})$ has the "total" grading.

We let $P_i \subset \mathrm{SL}(n)$ be the subgroup which is upper-triangular, except for the $(i+1, i)$ entry (a minimal parabolic). Then we let $X_{\mathbf{i}}$ be the variety

$$X_{\mathbf{i}} \cong P_{i_1} \times_B P_{i_2} \times_B \cdots \times_B P_{i_m}$$

This carries left and right torus actions.

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Smooth bimodules

We also have results for indecomposible bimodules, in particular, the proof of a conjecture of Rasmussen.

Theorem (W.-Williamson)

Let $w \in S_n$ be such that $\overline{BwB} \subset G$ is smooth. For some graded vector space V, of dimension $n-1$, we have an isomorphism of bigraded vector spaces:

$$HH_*(R_w) \cong R \otimes_{\mathbb{C}} H^*(\overline{BwB}) \cong R \otimes_{\mathbb{C}} \wedge^{\bullet} V$$

In particular, there is a series of integers k_i such that

$$\mathsf{hilb}(HH_*(R_w)) = \prod_{i=1}^{n-1} \frac{aq^{k_i} + a^{-1}q^{-k_i}}{q - q^{-1}}$$

The first isomorphism holds for non-smooth orbit closures using intersection cohomology instead. Skip example?

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An $n = 2$ example.

If $n = 2$, then there only two elements of the Weyl group. The corresponding orbit closures are $B \simeq S^1$ and $\mathrm{SL}(2) \simeq S^3$.

$$H^*(B) \cong \wedge^\bullet(\mathbb{C}\{-1\}) \qquad H^*(\mathrm{SL}(2)) \cong \wedge^\bullet(\mathbb{C}\{-3\})$$

Similarly, using the obvious free resolutions, we see that

$$HH_*(R) \cong R \otimes \wedge^\bullet(\mathbb{C}\{-2\}) \qquad HH_*(R \otimes_{R^{s_1}} R) \cong R \otimes \wedge^\bullet(\mathbb{C}\{-4\})$$

With a bit more work, you can check that the same thing happens in the case of SL(3). Left as an exercise to the bored.

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The proof

The first step is to identify $R_{\mathbf{i}}$ and $HH_*(R_{\mathbf{i}})$ with equivariant cohomology groups.

Proposition

For all $\mathbf{i}$, *we have* $R_{\mathbf{i}} \cong H^*_{T\times T}(X_{\mathbf{i}})$ *and* $HH_*(R_{\mathbf{i}}) \cong H^*_T(X_{\mathbf{i}})$, *with the respective* $R \otimes R$ *and* R *actions induced by the isomorphism* $R \cong H^*_T(pt)$.

By work of Rasmussen, $HH_*(R_{\mathbf{i}})$ is free. The usual theory of equivariant formality implies that

$$H^*_T(X_{\mathbf{i}}) \cong R \otimes_{\mathbb{C}} H^*(X_{\mathbf{i}})$$

The perspective of equivariant cohomology also lets us construct the differentials in the Rouquier complex geometrically, using pushforward and pullback maps for equivariant cohomology. Summarize?

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Gradings

Since $X_{\mathbf{i}}$ is smooth and equivariantly formal, we can separate the Hochschild and polynomial gradings.

Recall that by the Künneth theorem, $H_T^*(X_{\mathbf{i}}^T) \cong R \otimes_{\mathbb{C}} H^*(X_{\mathbf{i}}^T)$, so we can write the usual grading as a sum of "equivariant" and "topological" gradings.

By the equivariant formality, the pullback map $H_T^*(X_{\mathbf{i}}) \to H_T^*(X_{\mathbf{i}}^T)$ is injective. Using the above splitting, we can give $H_T^*(X_{\mathbf{i}})$ a similar bigrading.

Proposition

The isomorphism $H_T^*(X_{\mathbf{i}}) \cong HH^*(R_{\mathbf{i}})$ takes the "equivariant" and "topological" gradings to a linear combination of the obvious gradings on Hochschild homology.

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The isomorphism $H_T^*(X_{\mathbf{i}}) \cong HH^*(R_{\mathbf{i}})$ takes the "equivariant" and "topological" gradings to a linear combination of the obvious gradings on Hochschild homology.

Gradings

Since $X_{\mathbf{i}}$ is smooth and equivariantly formal, we can separate the Hochschild and polynomial gradings.

Recall that by the Künneth theorem, $H^*_T(X^T_{\mathbf{i}}) \cong R \otimes_{\mathbb{C}} H^*(X^T_{\mathbf{i}})$, so we can write the usual grading as a sum of "equivariant" and "topological" gradings.

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What's left

Summary: our model

- gives a geometric description of the Hochschild homology of indecomposable and Bott-Samelson Soergel bimodules.
- allows us to compute certain cases, as well as leverage for understanding general properties of this homology.
- gives a geometric description of the Rouquier complex.

What we hope for is

- a better understanding of homology in the Bott-Samelson space.
- a better understanding of intersection homology non-smooth cases.
- geometric methods for finding simplifications of the Rouquier complex, and more generally, applications to knot theory.

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