A geometric construction of colored HOMFLYPT homology

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Abstract. We show that the colored HOMFLYPT homology proposed by Mackaay, Stosic and Vaz is in fact a knot invariant categorifying the colored HOMFLYPT polynomial. Our method of construction is geometric, constructing this invariant in terms of the cohomology of various sheaves on algebraic groups, and giving the differentials and gradings a geometric interpretation.

1. INTRODUCTION

The *colored HOMFLYPT polynomial* is an invariant of links together with a labeling or "coloring" of each component with a positive integer; in particular, for knots, there is an invariant for each positive integer. Its most important properties are

- the colored invariant reduces to the usual HOMFLYPT polynomial when all labels are 1, and
- colored HOMFLYPT encapsulates all Reshetikhin-Turaev invariants for the link labeled with wedge powers of the standard representation of $\mathfrak{sl}_n$, just as the HOMFLYPT polynomial does for the standard representation alone.

In this paper we give a geometric construction of a categorification of this invariant, *colored HOMFLYPT homology.* Like the HOMFLYPT homology of of Khovanov and Rozansky [KR08, Kho07], this associates a triply graded vector space to each colored link such that the bigraded Euler characteristic is the colored HOMFLYPT polynomial.

Our initial construction and our proofs of invariance and categorification are algebrogeometric in nature, but we also show that this invariant has a purely combinatorial description of this invariant via bimodules. In fact, it coincides with that proposed from an algebraic perspective by Mackaay, Stosic and Vaz [MSV]. Thus, the main result of our paper has an entirely algebraic statement:

Theorem 1.1. *The colored HOMFLYPT homology defined in* [MSV] *is well-defined, a knot invariant, and its Euler characteristic coincides with the colored HOMFLYPT polynomial.*

Our definition also has the advantage of giving a categorification of essentially all algebraic objects involved in the definition of colored HOMFLY homology. Let us give a schematic diagram for the moving pieces here, with actual operations given by solid arrows, and categorifications given by dashed ones:

$$\begin{tikzcd}
 & \pi_\beta \mathbf{H}_N \pi_\beta \arrow[rrr, "\mathrm{Tr}_{JO}"] \arrow[dddd, dashed] & & & \mathbb{C}(q,t) \arrow[dddd, dashed] \\
 & & & \left\{ \begin{matrix} \text{MOY} \\ \text{graphs} \end{matrix} \right\} \arrow[ur, "\text{eval}"'] \arrow[dd, dashed] & \\
\left\{ \begin{matrix} \text{colored} \\ \text{braids} \end{matrix} \right\} \arrow[uur] \arrow[uurrrr, "\text{HOMFLYPT}"] \arrow[rr, "\beta \mapsto \hat{\beta}"] \arrow[ddr, "\Phi_\beta"'] \arrow[ddrrrr, "\mathcal{A}_2(\hat{\beta})"] & & \left\{ \begin{matrix} \text{colored} \\ \text{links} \end{matrix} \right\} \arrow[ur] \arrow[dr, "\mathcal{F}_L"] & & \\
 & & & D_{G_L}(X_L) \arrow[dr, "\mathbb{H}^*_{G_L}(X_L;-)"] & \\
 & D_{P_\beta \times P_\beta}(\mathrm{GL}(N)) \arrow[rrr, "\mathbb{H}^*_{(P_\beta)_\Delta}(\mathrm{GL}(N);-)"'] & & & \mathsf{g}^3\mathsf{Vect}
\end{tikzcd}$$

The top half of the diagram shows two different definitions of the colored HOMFLYPT polynomial:

- The path through {MOY graphs} is the description of the colored HOMFLYPT polynomial by [MOY98], by replacing a link diagram by a sum of trivalent weighted graphs, and then defining an evaluation function on such graphs.
- The path through $\pi_\beta \mathbf{H}_N \pi_\beta$ is described by [LZ]: to each closable colored braid β, we have an associated element of the Hecke algebra $\mathbf{H}_N$ where N is the colored braid index of β (the sum of the colorings of the strands). In fact, this element lies in a certain subalgebra $\pi_\beta \mathbf{H}_N \pi_\beta$ where π_β is a projection which depends on the coloring of β. The colored HOMFLY polynomial is obtained by applying a certain special trace Tr_{JO} defined by Ocneanu [Jon87] on $\mathbf{H}_N$.

In this paper, we show how to categorify both of these paths, as is schematically indicated in the bottom half of the diagram, and briefly summarized in Section 1.2.

- The left-most dashed arrow is a variant on the categorification of $\pi_\beta \mathbf{H}_N \pi_\beta$ by bi-equivariant sheaves over a parabolic P_β on $\mathrm{GL}(N)$.
- The central dashed arrow is an assignment to each MOY graph for a link diagram L of a simple perverse sheaf on a certain variety X_L which is equivariant for the action of a group G_L, both depending on the link diagram. These are the composition factors of a perverse sheaf assigned to the link itself.
- The right-most dashed arrow simply indicates the opposite to taking bigraded Euler characteristic of a tri-graded vector space with respect to one of its gradings.

We must also show that this diagram, including the dashed arrows "commutes." This follows directly from a result of the authors giving a similar construction of a Markov trace for the Hecke algebra of any semi-simple Lie group, shown in the paper [WWb].

As should be clear from the above, the techniques we use are those of algebraic geometry and geometric representation theory. While these are not familiar to the average topologist, we have striven to make this paper accessible to the novice, at least if they are willing accept a few deep results as black boxes. As a general rule, our actual calculations are simple and quite geometric in nature; however, we must cite rather serious machinery in order to show that these calculations are meaningful.

This geometric construction has several advantages over a purely combinatorial/algebraic approach. Several points which are difficult calculations from a combinatorial perspective are clear from geometry; for example, the complex attached to a single crossing is given by a page in a spectral sequence for the cohomology of a sheaf, which proves that $d^2 = 0$ (this proves Conjecture 1 of [MSV]).

1.1. Let us briefly indicate the geometric setting in which we work. This is discussed in considerably greater detail in Section 2.

Let X be an algebraic variety defined over a finite field $\mathbb{F}_q$. The machinery of étale sheaves may be used to associate to X a category $D^b(X;\overline{\mathbb{Q}_\ell})$, called the bounded derived category of $\overline{\mathbb{Q}_\ell}$-sheaves. (Here ℓ is a fixed prime number different from the characteristic of $\mathbb{F}_q$).

The category $D^b(X;\overline{\mathbb{Q}_\ell})$ is analogous to the bounded derived category of constructible sheaves on a complex algebraic variety. The advantage, however, of working over a finite field is that X is equipped with a Frobenius morphism $\mathrm{Fr} : X \to X$ and mixed sheaves on X come equipped with an isomorphism $\mathrm{Fr}^*\mathcal{F} \to \mathcal{F}$. It follows that a power of Frobenius acts on the stalks of $\mathcal{F}$ at points defined over some finite extension of $\mathbb{F}_q$.

The category $D^b(X;\overline{\mathbb{Q}_\ell})$ contains a remarkable abelian subcategory $P(X)$ of "mixed perverse sheaves." For us the most important feature of of $P(X)$ is that every object of $P(X)$ has a canonical "weight filtration" with semi-simple subquotients. As with any filtration, this leads to a spectral sequence

$$E_1^{p,q} = \mathbb{H}^{p+q}(\mathrm{gr}^W_{-p}\mathcal{F}) \Rightarrow \mathbb{H}^{p+q}(\mathcal{F}).$$

Each term on the left hand side carries an action of Frobenius, which may be used to give an additional grading to each page of the spectral sequence (which is trivial if X is proper). It follows that each page of the spectral sequence is *triply* graded.

We will also wish to consider the case of equivariant sheaves for the action of an algebraic groups, which poses some technical difficulties. While in principle this could be resolved by working in the category of stacks, we have found it less burdensome to give a careful definition of the mixed equivariant derived category from a more elementary perspective. For the sake of brevity, this has been done in a separate note [WWa].

1.2. In order to apply the above machinery to knot theory, we must define sheaves associated to a link. More precisely, as we discuss in Section 3, to any projection of a link L, we associate the natural graph Γ with vertices given by crossings and edges by arcs. To this graph, we associate a variety X_L over k together with the action of a reductive group G_L. The extra information provided by L allows us to construct a G_L-equivariant (shifted) perverse sheaf $\mathcal{F}_L \in D_{G_L}(X_L)$. We then show that $\mathcal{F}_L$ may be used to construct a series of knot invariants.

Theorem 1.2. *If L is the diagram of a closed braid, then all pages E_i for $i \geq 2$ of the spectral sequence computing $\mathbb{H}^*_{G_L}(X_L; \mathcal{F}_L)$ associated to the weight filtration is an invariant of L, up to an overall shift in the grading.*

Since we can endow $\mathbb{H}^*_{G_L}(X_L; -)$ of any mixed sheaf with the weight grading, which is preserved by all spectral sequence differentials, each page E_i for $i \geq 2$ is a triply-graded vector space, which is an invariant of the knot or link.

This description has a similar flavor to that of [KR08] or [BN05]: it begins by assigning a simple object to a single crossing, and then an algebraic rule for gluing crossings together (this process can be formalized as an object called a **canopolis** as introduced by Bar-Natan [BN05]). However, other papers, such as [Kho07] or [MSV] have used a description which depended strongly on the link diagram chosen being a closed braid. In order to show that our invariants coincide with those of [MSV], we must find a geometric description of this form.

Assume that β is a closable colored braid, $\hat{\beta}$ its closure and let N be the colored braid index (the sum of the colorings over the strands of the braid). Let P_β be the block upper triangular matrices inside G_N with the sizes of the blocks given by the coloring of the strands of β at the top and bottom (which coincide since β is closable). Using left and right multiplication, we obtain a natural $P_\beta \times P_\beta$ action on G_N. We let $(P_\beta)_\Delta$ be the diagonal subgroup, which acts on G_N be conjugation.

Theorem 1.3. *For each β, there is a $P_\beta \times P_\beta$-equivariant complex of sheaves Φ_β on $\mathrm{GL}(N)$ with a natural filtration, such that the associated spectral sequence computing $\mathbb{H}^*_{(P_\beta)_\Delta}(\mathrm{GL}(N); \Phi_\beta)$ is canonically isomorphic to the spectral sequence obtained from the weight filtration for $\mathbb{H}^*_{G_{\hat{\beta}}}(X_{\hat{\beta}}; \mathcal{F}_{\hat{\beta}})$.*

Furthermore, we have an isomorphism of the E_1 page of the spectral sequence for $\mathbb{H}^*_{P_\beta \times P_\beta}(\mathrm{GL}(N); \Phi_\beta)$ as a complex of bimodules over $H^*(BP_\beta)$ (which is naturally isomorphic to partially symmetric polynomials) to the complex of singular Soergel bimodules considered by Mackaay et al.

Since previous work of the authors [WW08] has related Hochschild homology to conjugation equivariant cohomology, we can identify our geometric knot invariant in terms of bimodules.

Theorem 1.4. *If L is a closed braid, then the E^2-page of our spectral sequence is the categorification of the colored HOMFLYPT polynomial proposed in* [MSV].

If all the labels on the components of L are 1, then this agrees with the triply-graded link homology as defined by Khovanov and Rozansky in [KR08].

2. MIXED AND EQUIVARIANT SHEAVES

2.1. This invariant is most naturally defined using the machinery of mixed equivariant sheaves. While this theory is rather deep and complicated in its full generality, we will only consider rather special cases. Thus we do not wish to give the reader the impression that a full understanding of this theory is genuinely necessary for reading our paper.

Instead, we intend to quickly summarize the properties of these sheaves which are necessary for us, and to indicate to the reader with a more serious interest in understanding the requisite algebraic geometry where the details can be found.

The important point underlying all this machinery is that cohomology of a complex algebraic variety (as well as most variations, such as equivariant cohomology, or intersection cohomology) has a natural grading, the **weight grading**. We call the sum of the weight grading and usual cohomological grading the **diagonal grading**. This grading is difficult to describe explicitly without using methods over characteristic p (as we will later), but is best understood by 2 simple properties

- The diagonal grading is preserved by cup products, by all pullback maps, and by all maps in long exact sequences (in fact, by all differentials in any Serre spectral sequence).
- This weight grading is trivial on projective varieties (i.e., the diagonal and usual gradings coincide).

Example 2.1 (The cohomology of $\mathbb{C}^*$). *If we write $\mathbb{CP}^1$ as the union of $\mathbb{C}$ and $\mathbb{CP}^1 - \{0\}$, then in the Mayer-Vietoris sequence, we have an isomorphism $H^2(\mathbb{CP}^1) \cong H^1(\mathbb{C}^*)$. Thus, $H^1(\mathbb{C}^*)$ has weight 1, that is, the cohomology of $\mathbb{C}^*$ is not pure.*

Since ultimately we plan to describe homological knot invariants using the equivariant cohomology of varieties, this grading will be necessary to give all the gradings we expect on our knot homology.

2.2. Sheaves and perverse sheaves.

We must use a generalization of this idea, the weight filtration on a mixed perverse sheaf. While there is a way to understand mixed perverse sheaves which only uses characteristic 0 methods (Saito's mixed Hodge modules [Sai86]; see the book of Peter and Steenbrink [PS08]), it is technically less demanding to consider sheaves on varieties in characteristic p. While this may sound daunting to the topologists in the audience, one can only see the difference from working with varieties over $\mathbb{C}$ in a few places, most notably reducing a difficult question of the structure of a certain sheaf to a simple point counting argument. In particular, every sheaf we consider will be constructed by applying familiar operations from algebraic geometry and has an analogue in terms of sheaves on complex varieties.

Let $q = p^e$ be a prime power. We consider throughout a finite field $\mathbb{F}_q$ with q elements and an algebraic closure $\mathbb{F}$ of $\mathbb{F}_q$. Unless we state otherwise all varieties and morphisms will be be defined over $\mathbb{F}_q$. Given a variety X we will write $X \otimes \mathbb{F}$ for its extension of scalars to $\mathbb{F}$.

We fix a prime number $\ell \neq p$ and let $\Bbbk$ denote the algebraic closure $\overline{\mathbb{Q}_\ell}$ of the field of ℓ-adic numbers. Throughout we fix a square root of q in $\Bbbk$ and denote it by $q^{1/2}$. Given a variety Y defined over $\mathbb{F}_q$ or $\mathbb{F}$ we denote by $D^b(Y)$ (resp. $D^+(Y)$) the bounded (resp. bounded below) derived category of constructible $\Bbbk$-sheaves on Y (see [Del77]). By abuse of language we also refer to objects in $D^b(X)$ or $D^+(X)$ as sheaves. Given a sheaf $\mathcal{F}$ on X we denote by $\mathcal{F} \otimes \mathbb{F}$ its extension of scalars to a sheaf on $X \otimes \mathbb{F}$. Given a sheaf $\mathcal{F}$ on X we abuse notation and write

$$\mathbb{H}^*(\mathcal{F}) := \mathbb{H}^*(X \otimes \mathbb{F}, \mathcal{F} \otimes \mathbb{F}) = \mathbb{H}^*(\mathcal{F} \otimes \mathbb{F}).$$

We *never* consider hypercohomology before extending scalars.

On the category $D^b(X)$, we have the usual system of functors

- bifunctors of
  - sheaf homomorphisms $\mathscr{H}om : D^b(X)^{op} \times D^b(X) \to D^b(X)$
  - tensor product $\otimes : D^b(X) \times D^b(X) \to D^b(X)$ (sometimes denoted $\overset{L}{\otimes}$),
- the Verdier duality functor $\mathbb{D} : D^b(X) \to D^b(X)^{op}$,
- for each map $f : X \to Y$, we have

- Verdier dual pushforward functors $f_*, f_! : D^b(X) \to D^b(Y)$ (usually denoted Rf_* and $Rf_!$)
- Verdier dual pullback functors $f^*, f^! : D^b(Y) \to D^b(X)$.

In $D^b(X)$ we have the full abelian subcategory $P(X)$ of **perverse sheaves** (see [BBD82]). We will call a sheaf $\mathcal{F}$ **shifted perverse** if $\mathcal{F}[n]$ is perverse for some $n \in \mathbb{Z}$.

2.3. The Frobenius and its action on sheaves. Given any variety X defined over $\mathbb{F}_q$ we have the Frobenius morphism

$$\mathrm{Fr}_q : X \to X$$

which for affine $X \subset \mathbb{A}^n$ is given by $(x_1, \dots, x_n) \mapsto (x_1^q, \dots, x_1^q)$. The fixed points of $\mathrm{Fr}_{q^n} := (Fr_q)^n$ are precisely $X(\mathbb{F}_{q^n})$, the points of X defined over $\mathbb{F}_{q^n}$.

Given any $\mathcal{F} \in D^b(X)$ we have an isomorphism

$$F_q^* : \mathrm{Fr}_q^* \mathcal{F} \xrightarrow{\sim} \mathcal{F}.$$

Thus we have an induced action of $F_{q^n}^* := (F_q^*)^n$ on the stalk of $\mathcal{F}$ at any point $x \in X(\mathbb{F}_{q^n})$. By considering the eigenvalues of the action of $F_{q^n}^*$ on all the stalks of $\mathcal{F}$ at all points $x \in X(\mathbb{F}_{q^n})$ for all $n \geq 1$ one defines full subcategories $D_m^b(X)$, $D_{\leq w}^b(X)$ and $D_{\geq w}^b(X)$ (for $w \in \mathbb{Z}$) of **mixed sheaves**, sheaves of **weight** $\leq w$ and **weight** $\geq w$ respectively (see Chapter 5 of [BBD82], [Del80] or the first chapter of [KW01]). An object is called **pure of weight** i if it lies in both $D_{\leq i}^b(X)$ and $D_{\geq i}^b(X)$.

Given any mixed sheaf $\mathcal{F}$ on X all eigenvalues $\alpha \in \Bbbk$ of Fr_q^* on $\mathbb{H}^*(\mathcal{F})$ are algebraic integers such that all complex numbers with the same minimal polynomial have the same complex norm, which by abuse of notation, we denote $|\alpha|$. Let $\mathbb{H}_\alpha^*(\mathcal{F}) \subset \mathbb{H}^*(\mathcal{F})$ be the generalized eigenspace of α, and let

$$\mathbb{H}^{*,i}(\mathcal{F}) := \bigoplus_{|\alpha| = q^{i/2}} \mathbb{H}_\alpha^*(\mathcal{F}).$$

This perspective relates to our previous discussion as follows: The constant sheaf on X is mixed, and its hypercohomology is the étale cohomology of X. The i-th graded component of $H^*(X; \Bbbk)$ for the diagonal grading is $H^{*;i}(X; \Bbbk)$. So, our previous discussion was just a reflection of some of the properties of the Frobenius action on the cohomology of algebraic varieties. While this action is rather hard to understand from the perspective of characteristic 0 geometry, the extra structure it gives us is precisely the advantage of working over characteristic p, and is essential in producing the gradings on knot homology we require.

If $X_0 = \operatorname{Spec} \mathbb{F}_q$ then a perverse sheaf on X_0 is the same as a finite dimensional $\Bbbk$-vector space together with a continuous action of the absolute Galois group of $\mathbb{F}_q$. In particular we have the **Tate sheaf** $\underline{\Bbbk}(1)$ which, under the above equivalence, corresponds to $\Bbbk$ with action of F_q^* given by q^{-1}. Recall that we have fixed a square root $q^{1/2}$ of q in $\Bbbk$ allowing us to define the **half Tate sheaf** $\underline{\Bbbk}(1/2)$, with F_q^* acting by $q^{-1/2}$.

Given any X with structure morphism $X \xrightarrow{a} \operatorname{Spec} \mathbb{F}_q$ and any sheaf $\mathcal{F}$ on X we define

$$\mathcal{F}(m/2) := \mathcal{F} \otimes a^* \underline{\Bbbk}(1/2)^{\otimes m}.$$

The following notation will prove useful:

$$\mathcal{F}\langle d \rangle = \mathcal{F}[d](d/2).$$

Note that $\langle d \rangle$ preserves weight.

The most important fact about mixed sheaves for our purposes is that every mixed perverse sheaf $\mathcal{F}$ on X admits a unique increasing filtration W, called the **weight filtration**, such that, for all i,

$$\mathrm{gr}_i^W \mathcal{F} := W_i\mathcal{F}/W_{i-1}\mathcal{F}$$

is pure of weight i.

In fact, after extension of scalars to the algebraic closure, the extensions in this filtration are the only way that mixed perverse sheaves can fail to be semi-simple.

Theorem 2.2 (Gabber; [BBD82] Théorème 5.3.8)**.** *If $\mathcal{F}$ is a pure perverse sheaf on X then $\mathcal{F}\otimes\mathbb{F}$ is semi-simple.*

2.4. The function-sheaf dictionary. The eigenvalues of Frobenius on stalks are also valuable for analyzing the structure of a given perverse sheaf. To any mixed perverse sheaf $\mathcal{F}$ (or more generally, any mixed sheaf) one may associate a family of functions on $X(\mathbb{F}_{q^n})$ given by the supertrace of the Frobenius on the stalks of the cohomology sheaves at those points:

$$\begin{aligned}[\mathcal{F}]_n : X(\mathbb{F}_{q^n}) &\to \Bbbk\\ x &\mapsto \mathrm{Tr}(F_{q^n}^*, \mathcal{F}_x) := \sum(-1)^j \mathrm{Tr}(F_{q^n}^*, \mathcal{H}^j(\mathcal{F}_x)).\end{aligned}$$

Proposition 2.3. *These functions give an injective map from the Grothendieck group of the category of mixed perverse sheaves to the abelian group of functions on $X(\mathbb{F}_{q^n})$ for all n. That is, if $\mathcal{F}$ and $\mathcal{G}$ are semi-simple and $[\mathcal{F}]_n = [\mathcal{G}]_n$ for all n then $\mathcal{F}$ and $\mathcal{G}$ are isomorphic.*

Proof. The fact that these functions give a map of Grothendieck groups is just that all maps in the long exact sequence must respect the action of the Frobenius, so the supertrace is additive under extensions.

Injectivity follows from [KW01, Theorem 12.1]. □

This reduces the calculation of the constituents of a weight filtration to a problem of computing $[\mathcal{F}]_n$ for simple perverse sheaves, followed by linear algebra. Indeed, suppose that $\mathcal{F}, \mathcal{G} \in D(X_o)$ are such that $[\mathcal{F}]_n$ and $[\mathcal{G}]_n$ agree with $\mathcal{G}$ semi-simple. As $[\mathcal{F}]_n = \sum[\mathrm{gr}_i^W \mathcal{F}]_n$ for all n we conclude that $\mathrm{gr}_i^W \mathcal{F}$ is isomorphic to the largest direct summand of $\mathcal{G}$ of weight i.

2.5. The chromatographic complex. We want to explain how to move between the weight filtration and a complex, which we term **the chromatographic complex**, composed of its pure constituents.

Let us write

$$\mathrm{gr}_{i,i+1}^W \mathcal{F} = W_{i+1}\mathcal{F}/W_{i-1}\mathcal{F}.$$

For all i, we have an exact sequence of perverse sheaves

$$0 \longrightarrow \mathrm{gr}_i^W \mathcal{F} \longrightarrow \mathrm{gr}_{i,i+1}^W \mathcal{F} \longrightarrow \mathrm{gr}_{i+1}^W \mathcal{F} \longrightarrow 0.$$

Remember that an exact sequence of perverse sheaves is the same thing as an exact triangle in the ambient category. Hence the above exact sequence contains the information of a map

$$\mathrm{gr}_{i+1}^W \mathcal{F}[-(i+1)] \to \mathrm{gr}_i^W \mathcal{F}[-i].$$

We can do this for all i and obtain a sequence

$$\cdots \to \mathrm{gr}_{i+1}^W \mathcal{F}[-(i+1)] \to \mathrm{gr}_i^W \mathcal{F}[-i] \to \mathrm{gr}_{i-1}^W \mathcal{F}[-(i-1)] \to \cdots \tag{1}$$

Definition/Theorem 2.4. *The sequence of maps (1) is a complex. We call this the* **local chromatographic complex** *of $\mathcal{F}$.*

Applying hypercohomology, we obtain a complex of vector spaces

$$\cdots \to \mathbb{H}^*(\mathrm{gr}^W_{i+1} \mathcal{F}[-(i+1)]) \to \mathbb{H}^*(\mathrm{gr}^W_i \mathcal{F}[-i]) \to \mathbb{H}^*(\mathrm{gr}^W_{i-1} \mathcal{F}[-(i-1)]) \to \cdots$$

which we call the **(global) chromatographic complex** *of $\mathcal{F}$.*

Proof. This follows from the octahedral axiom of triangulated categories. □

We note that $\mathbb{H}^*(\mathrm{gr}^W_i \mathcal{F}[-i])$ is, in fact, naturally bigraded by the cohomological and the weight grading, so the cohomology of this complex is *triply*-graded. As before, we let $\mathbb{H}^{i;j}(X;\mathcal{F})$ be the portion of the ith cohomology of weight $j-i$.

In fact, the chromatographic complex makes sense for any object in $D^+(X)$. Any such object can written as a complex $\mathbf{F}^\bullet$ of mixed perverse sheaves. Thus, we can apply the weight filtration term-wise and obtain a local chromatographic bicomplex

$$\cdots \to \mathrm{gr}^W_{i+1} \mathbf{F}^\bullet[-(i+1)] \to \mathrm{gr}^W_i \mathbf{F}^\bullet[-i] \to \mathrm{gr}^W_{i-1} \mathbf{F}^\bullet[-(i-1)] \to \cdots \tag{2}$$

Remark 1. There is a subtle point here; we think of the weight filtration on the terms of our complex in a naive way, without shifting the weight filtration to account for placement in the complex, as is the usual convention.

By Gabber's theorem, $\mathrm{gr}^W_i \mathbf{F}^\bullet$ is semi-simple, so up to homotopy, we can replace $\mathrm{gr}^W_i \mathbf{F}^\bullet$ by its cohomology.

Furthermore, since all maps between perverse sheaves strictly preserve the weight filtration, taking the weight filtration commutes with taking cohomology. Thus, up to "vertical" homotopy, the complex above is unchanged by replacing $\mathbf{F}^\bullet$ by a quasi-isomorphic complex.

Definition 2.5. *The total complex of the bicomplex (2) is the local chromatographic complex of $\mathbf{F}^\bullet$, and its hypercohomology is the global chromatographic complex. As we noted above, this is well defined up to homotopy (and in fact, all such homotopies are simply stripping off a trivial summand).*

Proposition 2.6. *The global chromatographic complex is preserved by proper pushforward.*

Proof. The weight filtration is preserved by proper pushforward, as is hypercohomology. □

As usual, the weight filtration gives a spectral sequence for any functor F applied to $\mathcal{F}$, of the form

$$E_1^{i,j} = R^{i+j}F(\mathrm{gr}^W_{-j} \mathcal{F}) \Rightarrow E_\infty^{i+j} = R^{i+j}F(\mathcal{F})$$

A simple diagram chase shows that

Proposition 2.7. *The differentials of the complex given by F applied to the local chromatographic complex coincide with the differentials on E_1 for this sequence.*

Definition 2.8. *We call the spectral sequence obtained when $F = \mathbb{H}^*(-)$ the* **chromatographic spectral sequence**.

Corollary 2.9. *If we let $E_*^{*,*}$ be the chromatographic spectral sequence, then all differentials preserve the diagonal grading on hypercohomology. Furthermore, we have*

- $E_1^{i,j} = \mathbb{H}^{i+j}(\mathrm{gr}^W_{-j} \mathcal{F})$ *is the global chromatographic complex.*

- *E_2 is the cohomology of the chromatographic complex.*
- $E_\infty^{i+j} \cong \mathbb{H}^{i+j}(\mathcal{F})$.

2.6. Equivariant sheaves and their derived category. We have thus far discussed the theory of perverse sheaves on schemes, but we will require a slight generalization of schemes which includes the quotient of a scheme X by the action of an algebraic group G, which can typically be understood as G-equivariant geometry on X.

This quotient can be understood as a stack, but the theory of perverse sheaves on stacks is not straightforward, and it proved more suitable to give a treatment of the equivariant derived category similar to that of Bernstein and Lunts [BL94], but with an eye to working over characteristic p with the action of the Frobenius (that is "in the mixed setting"). We have done this in a separate note [WWa].

The result is the **bounded below equivariant derived category** $D_G^+(X)$ and its subcategory $D_G^b(X)$ of bounded sheaves for a variety X acted on by an affine algebraic group G. The resulting formalism is essentially identical to that of Bernstein and Lunts. We now summarize the essential points.

We have a forgetful functor

$$\mathrm{For} : D_G^+(X) \to D^+(X)$$

which preserves the subcategories of bounded sheaves and, given any $\mathcal{F} \in D_G^+(X)$, the cohomology sheaves of $\mathrm{For}(\mathcal{F})$ are locally constant along the G-orbits on X.

Given an equivariant map $f : X \to Y$ of G-varieties we have functors

$$f_*, f_! : D_G^+(X) \to D_G^+(Y)$$

and

$$f^*, f^! : D_G^+(Y) \to D_G^+(X)$$

for equivariant maps $f : X \to Y$ of G-varieties. These functors commute with the forgetful functor.

If $H \subset G$ is a closed subgroup and X is a G-space we have an adjoint pair $(\mathrm{res}_H^G, \mathrm{ind}_H^G)$ of restriction and induction functors

$$\mathrm{res}_H^G : D_G^+(X) \to D_H^+(X) \quad \mathrm{ind}_H^G : D_H^+(X) \to D_G^+(X).$$

These preserve the subcategories of bounded sheaves, and one has an isomorphism $\mathrm{res}_{\{1\}}^G \cong \mathrm{For}$.

More generally, given a map $\phi : H \to G$, a G-variety X, an H-variety Y and a ϕ-equivariant map $m : X \to Y$ we have an adjoint pair $({}_H^G m^*, {}_H^G m_*)$ of functors

$${}_H^G m^* : D_H^+(Y) \to D_G^+(X) \text{ and } {}_H^G m_* : D_G^+(X) \to D_H^+(Y).$$

As a special case, we have ${}_H^G id^* = \mathrm{res}_H^G, {}_H^G id_* = \mathrm{ind}_H^G$. The functor ${}_H^G m^*$ preserves the subcategory of bounded sheaves, but this is not true in general for ${}_H^G m_*$. In fact, this is the reason that we are forced to consider unbounded sheaves.

If $G = G_1 \times G_2$ and G_1 acts freely on X with quotient X/G_1 one has an equivalence

$$D_G^+(X) \cong D_{G_2}^+(X/G_1)$$

which restricts to an equivalence between the subcategories of bounded sheaves. If we let $\phi : G_1 \times G_2 \to G_2$ denote the projection then the quotient map $X \to X/G_1$ is ϕ-equivariant and the above equivalence is realized by ${}_{G_1 \times G_2}^{G_2} m^*$ and ${}_{G_1 \times G_2}^{G_2} m_*$.

Many notions carry over immediately using the forgetful functor $\mathrm{For} : D^+_G(X) \to D^+(X)$. For example, we call an object $\mathcal{F}$ in $D^+_G(X)$ **perverse** if and only if $\mathrm{For}\,\mathcal{F}$ is perverse.

However if X is defined over $\mathbb{F}_q$, then we can also incorporate the action of the Frobenius. In particular, perverse objects in $D^+_G(X)$ still have weight filtrations, which are preserved by the restriction functor and we can extend Proposition 2.3 to the equivariant setting using the forgetful functor as long as our group is connected.

3. DESCRIPTION OF THE INVARIANT

We start by recalling the steps involved in our categorification, beginning with a braid-like diagram L of an oriented colored link:

- To L we associate a reductive group G_L together with a G_L-variety X_L, which only depends on the graph Γ obtained from the diagram L by forgetting under- and overcrossings.
- The extra data contained in L allows us to define a G_L-equivariant sheaf $\mathcal{F}_L$ on X_L.
- This sheaf $\mathcal{F}_L$ has a chromatographic spectral sequence converging to the G_L-equivariant hypercohomology of $\mathcal{F}_L$.
- Each page of this spectral sequence is a knot-invariant and the E_2 page categorifies the colored HOMFLYPT polynomial.

In this section we discuss the first three steps.

3.1. First let us fix some notation. We fix throughout a chain of vector spaces $0 \subset V_1 \subset V_2 \subset V_3 \subset \cdots$ over $\mathbb{F}_q$ such that $\dim V_i = i$ for all i. Let

$$G_{i_1,\dots,i_n} := \mathrm{GL}(i_1) \times \cdots \times \mathrm{GL}(i_n),$$

and let $P_{i_1,\dots,i_n}$ be the block upper-triangular matrices with blocks $\{i_1, \dots, i_n\}$. We may identify $P_{i_1,i_2,\dots,i_n}$ with the stabilizer in $G_{i_1+\cdots+i_n}$ of the standard partial flag

$$\{0 \subset V_{i_1} \subset V_{i_1+i_2} \subset \cdots \subset V_{i+1+\cdots+i_n}\}.$$

Let L be a diagram of an oriented tangle with marked points, with no marked points occuring at a crossing. Let Γ be the oriented graph obtained by forgetting over- and undercrossings. We deal with the exterior ends of the tangle in a somewhat unconventional manner; we do not think of them as vertices in the graph, so we think of the arcs connecting to the edge as connecting to 1 or 0 vertices. By adding marked points to L if necessary, we may assume that every component of Γ contains at least one vertex.

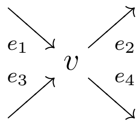
Recall that, to the diagram Γ we wish to associate a variety X_L acted on by an algebraic group G_L. Let us write $\mathcal{E}(\Gamma)$ and $\mathcal{V}(\Gamma)$ for the edges and vertices of Γ respectively. Given an edge $e \in \mathcal{E}(\Gamma)$ write G_e for G_i, where i is the label on e. Similarly, given $v \in \mathcal{V}(\Gamma)$ write G_v for G_i where i is the sum of the labels on the incoming vertices at v. We define

$$X_L := \prod_{v \in \mathcal{V}(\Gamma)} G_v$$

and

$$G_L := \prod_{e \in \mathcal{E}(\Gamma)} G_e.$$

It remains to describe how G_D acts on X_D. Locally, near any crossing, Γ is isotopic to a graph of the form



we will call e_1 and e_2 **upper** and e_3 and e_4 **lower** vertices with respect to the vertex v. Whenever a vertex v lies on an edge e we define an inclusion map $i_e : G_e \to G_v$ which is the identity if v corresponds to a marked point, and is the composition

$$\begin{aligned} G_i \hookrightarrow G_{i,j} \hookrightarrow G_{i+j} \quad & \text{if } e \text{ is upper,} \\ G_i \hookrightarrow G_{j,i} \hookrightarrow G_{i+j} \quad & \text{if } e \text{ is lower.} \end{aligned}$$

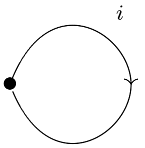
That is, G_e is included as the upper left or lower right block matrices in G_v, according to whether e is upper or lower.

We now describe how G_L acts on X_L by describing the action componentwise. Let $g \in G_e$ and $x \in G_v$. We have

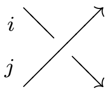
$$g \cdot x = \begin{cases} x & \text{if } v \text{ does not lie on } e, \\ x i_e(g)^{-1} & \text{if } e \text{ is outgoing at } v, \\ i_e(g) x & \text{if } e \text{ is incoming at } v. \end{cases}$$

Example 3.1. *Here are two examples of X_L and G_L:*

- *If L is the natural diagram of the unknot labeled i with one marked point*



  we have $X_L = G_L = G_i$ and G_L acts on X_L by conjugation.
- *Let L be the a diagram of an (i, j)-crossing:*

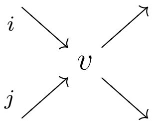


  Here $X_L = G_{i+j}$ and $G_D = G_i \times G_j \times G_j \times G_i$ and (a, b, c, d) acts on $x \in G_{i+j}$ by

$$\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} x \begin{pmatrix} c^{-1} & 0 \\ 0 & d^{-1} \end{pmatrix}.$$

3.2. In this subsection we describe the sheaf $\mathcal{F}_L$ on X_L.

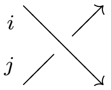
We first discuss the case of a single (i, j)-crossing:



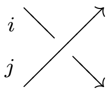
As we have seen $X_L = G_{i+j}$. Consider the big Bruhat cell

$$U := \{g \in G_{i+j} \mid V_i \cap gV_j = 0\} \tag{3}$$

and let $j : U \hookrightarrow G_{i+j}$ denote its inclusion. As U is an orbit under $P_{i,j} \times P_{j,i}$ it is certainly G_L-invariant. We now define $\mathcal{F}_v = \mathcal{F}_L \in D_{G_L}(X_L)$ as follows:



$$\mapsto \quad j_*\underline{\Bbbk}_U\langle ij\rangle$$



$$\mapsto \quad j_!\underline{\Bbbk}_U\langle ij\rangle$$

As U is the complement of a divisor in G_{i+j} both these sheaves are (shifted) perverse.

We now consider the case of a general diagram L of an oriented colored tangle. After forgetting equivariance $\mathcal{F}_L$ is simply the exterior product of the above sheaves associated to each crossing. To take care of the equivariant structure we need to proceed a little more carefully.

Let L be the diagram of an oriented colored tangle and Γ its underlying graph. Let L' be the diagram obtained from L by cutting each strand connecting two vertices in Γ (so that L' is a disjoint union of (i,j)-crossings). Let Γ' be the graph corresponding to L'. Obviously we have $X_L = X'_L$. Note also that for every e with two vertices in Γ, we have two edges, which we denote e_1 and e_2 in Γ'. We have a natural map $G_L \to G'_L$ which is the identity on factors corresponding to edge strands, and is the diagonal $G_e \to G_{e_1} \times G_{e_2}$ on the remaining factors.

We define

$$\mathcal{F}_{L'} := \operatorname{res}^{G}_{G'}\Big(\boxtimes_{v\in\mathcal{V}(\Gamma')} \mathcal{F}_v\Big) \in D_{G_L}(X_L).$$

Of course, this sheaf depends on the link diagram used; different diagrams correspond to sheaves on different spaces. Instead, we will studying the hypercohomology of these sheaves, and the corresponding chromatographic spectral sequence.

Definition 3.2. *We let $\mathcal{A}_i(L)$ denote the ith page of the chromatographic spectral sequence (as given by Definition 2.8) for $\mathcal{F}_L$. This is triply graded, where by convention subquotients of $\mathbb{H}^{j-\ell;\frac{j-k}{2}}(\operatorname{gr}^W_\ell \mathcal{F}_L)$ lies in $\mathcal{A}_i^{j;k;\ell}(L)$.*

Remark 2. These grading conventions may seem strange, but they are an attempt to match those already in use in the field. These conventions are almost those of [MSV], though we will not match perfectly since we have different grading shifts in our definition of the complex for a single crossing. We hope the reader finds these choices defensible on grounds of geometric naturality. This simply changes the shift we must apply to our invariant do assure it is a true knot invariant.

It is these spaces for $i > 1$ which we intend to show are knot invariants (up to shift).

3.3. Braids and sheaves on groups. As we mentioned in Section 1, in the special case of a braid β, there is a different perspective on this construction.

So let β be a colored braid on n strands with labels $\mathbf{n} = (i_1, i_2, \ldots, i_n)$ and underlying labeled graph Γ. Let $N = \sum_{j=1}^n i_j$ denote the colored braid index. We fix an ordering of the vertices $v_1, v_2, \ldots, v_p$ corresponding to an expression for β in the standard generators of the braid group. In the previous section we described how to associate to β a group G_β and a G_β-variety X_β.

We can decompose G_β as

$$G_\beta = G_\beta^+ \times G_\beta^\iota \times G_\beta^-$$

where G_β^+, G_β^ι and G_β^- denote the factors of G_β corresponding to incoming, interior and outgoing edges of Γ respectively.

In what follows we will describe an action of $G_\beta^+ \times G_\beta^-$ on G_N and a map

$$m : X_\beta \to G_N$$

equivariant with respect to the natural projection $\phi : G_\beta \to G_\beta^+ \times G_\beta^-$. This map will will allow us to reduce questions about the sheaf $\mathcal{F}_\beta$ to questions about a sheaf Φ_β on G_N.

We start by describing an embedding $\alpha_v : G_v \to G_N$ corresponding to each vertex $v \in \Gamma$. Let us fix a basis $e_1, \ldots, e_N$ of V_N and let $W_1, W_2, \ldots, W_n$ be vector spaces (again with fixed bases) of dimensions $i_1, i_2, \ldots, i_n$ respectively. Given any permutation $w \in S_n$ we have an isomorphism

$$h_w : W = \bigoplus_{j=1}^{n} W_j \xrightarrow{\sim} V$$

by mapping the basis vectors of $W_{w^{-1}(1)}$ to the first $w^{-1}(1)$ basis vectors of V, the basis vectors of $W_{w^{-1}(2)}$ to the next $w^{-1}(2)$ basis vectors etc. For any braid β, we have an induced permutation, and by abuse of notation, we let h_β be the map corresponding to this permutation.

Now choose a vertex v in Γ, let e' and e'' denote the two incoming edges, which are in the strands connected to the j'th and j''th incoming vertex respectively, so $i_{j'}, i_{j''}$ are the labels on e' and e''. Because we have ordered the vertices of Γ, we may factor β into braids $\alpha_v \cdot \beta_v \cdot \omega_v$ with β_v consisting of a simple crossing corresponding to v. The procedure described in the previous paragraph yields an embedding $W_{j'} \oplus W_{j''} \hookrightarrow W \xrightarrow{h_{\alpha_v}} V_N$. This induces an embedding

$$\iota_v : G_v \hookrightarrow G_N$$

We let braids on n strands act on sequences of n elements on the right by the usual association of a permutation to each braid. We may then identify

$$G_\beta^+ \cong G_{\mathbf{n}}$$
$$G_\beta^- \cong G_{\mathbf{n}\beta}$$

and therefore obtain an action of $G_\beta^+ \times G_\beta^-$ on G_N by left and right multiplication. We let $P_\beta^+ = P_{\mathbf{n}}, P_\beta^- = P_{\mathbf{n}\beta}$. We denote by $\phi : G_\beta \to P_\beta^+ \times P_\beta^-$ be the composition of the natural projection with the inclusion $G_\beta^\pm \hookrightarrow P_\beta^\pm$.

Consider the map

$$m : X_\beta \to G_N$$
$$(g_{v_1}, \ldots, g_{v_p}) \mapsto \iota_{v_1}(g_{v_1})\iota_{v_2}(g_{v_2}) \ldots \iota_{v_p}(g_{v_p})$$

It is easy to see that this map is equivariant with respect to ϕ.

Definition 3.3. *Let* $\Phi_\beta = {}^{P_\beta^+ \times P_\beta^-}_{G_\beta} m_* \mathcal{F}_\beta$.

This definition is useful, since it is compatible with braid multiplication. We have a diagram of equivariant maps of spaces

$$\begin{tikzcd}
G_N & & \\
& G_N \times G_N \arrow[ul, "\pi_1"'] \arrow[dl, "\pi_2"] \arrow[r, "\mu"] & G_N \\
G_N & &
\end{tikzcd}$$

We have a natural functor

$$- \star - : D^b_{P_{\mathbf{n}} \times P_{\mathbf{n}\beta}}(G_N) \times D^b_{P_{\mathbf{n}\beta} \times P_{\mathbf{n}\beta\beta'}}(G_N) \to D^b_{P_{\mathbf{n}} \times P_{\mathbf{n}\beta\beta'}}(G_N)$$

$$\mathcal{F}_1 \star \mathcal{F}_2 \cong {}^{P_{\mathbf{n}} \times P_{\mathbf{n}\beta\beta'}}_{P_{\mathbf{n}} \times P_{\mathbf{n}\beta} \times P_{\mathbf{n}\beta\beta'}} \mu_* \left(\mathrm{res}^{P_{\mathbf{n}} \times P^2_{\mathbf{n}\beta} \times P_{\mathbf{n}\beta\beta'}}_{P_{\mathbf{n}} \times P_{\mathbf{n}\beta} \times P_{\mathbf{n}\beta\beta'}} \mathcal{F}_1 \boxtimes \mathcal{F}_2 \right).$$

Theorem 3.4. *We have a canonical isomorphism* $\Phi_\beta \star \Phi_{\beta'} \cong \Phi_{\beta\beta'}$.

Proof. Immediate from the definition of Φ. ☐

As G^ι_β acts freely on X_β, and we may factor m as

$$X_\beta \to X_\beta / G^\iota_\beta \to G_N.$$

One may verify that the second map is the composition of an affine bundle along which $\mathcal{F}_\beta$ is smooth, and a proper map. It follows that ${}^{P^+_\beta \times P^-_\beta}_{G_\beta} m_*$ preserves the weight filtration on $\mathcal{F}_\beta$. Hence the chromatographic spectral sequences for $\mathcal{F}_\beta$ and Φ_β are isomorphic.

Note that if β is closable, then $\mathbf{n}\beta = \mathbf{n}$, and $P^\pm_\beta$ have the same image in the group, and thus are canonically isomorphic. Let $(P_\beta)_\Delta$ be the diagonal. Let $\hat{\beta}$ be the colored link diagram given by the closure of β.

Theorem 3.5. *We have a canonical isomorphism between*

- *the chromatographic spectral sequence of* $\mathcal{F}_{\hat{\beta}}$ *as a* $G_{\hat{\beta}}$*-sheaf and*
- *the chromatographic spectral sequence of* Φ_β *as a* $(P_\beta)_\Delta$*-sheaf.*

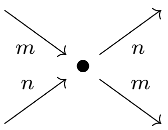
Proof. Since P_* and G_* are homotopy equivalent, the functor $\mathrm{res}^{P_*}_{G_*}$ is fully faithful, so we may work with $(G_\beta)_\Delta$-equivariant cohomology. We have already observed that the weight filtrations on Φ_β and $\mathcal{F}_\beta$ agree. Thus the equivariant chromatographic spectral sequences of $\mathrm{res}^{G_\beta}_{\phi^{-1}(H)} \mathcal{F}_\beta$ and $\mathrm{res}^{G^+_\beta \times G^-_\beta}_{H} \Phi_\beta$ are canonically isomorphic for any subgroup $H \subset G^+_\beta \times G^-_\beta$.

On the other hand, we have a canonical identification $G_{\hat{\beta}} \cong \phi^{-1}\big((G_\beta)_\Delta\big)$, and $X_\beta = X_{\hat{\beta}}$, with $\mathcal{F}_{\hat{\beta}} = \mathrm{res}^{G_\beta}_{G_{\hat{\beta}}} \mathcal{F}_\beta$. The result follows. ☐

4. ANALYZING AN (m, n)-CROSSING

4.1. In this section we work out all the details for an (m, n)-crossing. This will be of use in expressing the invariant in terms of bimodules.

We consider an (m, n)-crossing. Its underlying graph is



and the variety in question is G_{m+n} acted on by $P_{m,n} \times P_{n,m}$ by left and right multiplication: $(p,q) \cdot g = pgq^{-1}$ for $g \in G_{n+m}$ and $(p,q) \in P_{m,n} \times P_{n,m}$. The orbits under this action are

$$\mathcal{O}_i = \{g \in G_{m+n} \mid \dim V_m \cap gV_n = i\} \text{ for } 0 \leq i \leq \min(n,m).$$

Clearly $\mathcal{O}_j \subset \overline{\mathcal{O}_i}$ if and only if $j > i$. For all $0 \leq i \leq \min(n,m)$ we denote by $f_i : \mathcal{O}_i \hookrightarrow G_{n+m}$ the inclusion of the orbit.

For each orbit $\mathcal{O}_i$ we have the corresponding intersection cohomology complex. It will prove natural to normalize them by requiring

$$\mathbf{IC}(\overline{\mathcal{O}_i})_{|\mathcal{O}_i} \cong \underline{\Bbbk}_{\mathcal{O}_i}\langle nm - i^2\rangle.$$

Under this normalization each $\mathbf{IC}(\overline{\mathcal{O}_i})$ is pure of weight 0.

We first describe resolutions for the closures $\overline{\mathcal{O}_i} \subset G_{m+n}$. Consider the variety

$$\widetilde{\mathcal{O}_i} = \{(W,g) \in \mathrm{Gr}_i^m \times G_{m+n} \mid W \subset V_m \cap gV_n\}.$$

We have an action of $P_{m,n} \times P_{n,m}$ on $\widetilde{\mathcal{O}_i}$ given by $(p,q) \cdot (W,g) = (pW, pgq^{-1})$. The second projection induces an equivariant map:

$$\pi_i : \widetilde{\mathcal{O}_i} \to \overline{\mathcal{O}_i}.$$

Proposition 4.1. *This is a small resolution of singularities.*

Proof. The morphism π_i is patently an isomorphism over $\mathcal{O}_i$. Since $\mathcal{O}_i$ is exactly the subset of G_{n+m} where the induced map $V_n \to V/V_m$ has rank $n-i$, we have that $\mathcal{O}_i$ has the same codimension in G_{m+n} as the space of rank $n-i$ matrices in G_n, which is i^2. Hence, for $j < i$, $\mathcal{O}_i$ is of codimension $i^2 - j^2$ in $\overline{\mathcal{O}_j}$. Over any $x \in \mathcal{O}_j$ the fiber is the Grassmannian Gr_i^j. Thus

$$2\dim \pi_i^{-1}(x) = 2i(j-i) < (j+i)(j-i) = \mathrm{codim}_{\overline{\mathcal{O}_i}}\mathcal{O}_j. \qquad \square$$

Corollary 4.2. $\mathbf{IC}(\overline{\mathcal{O}_i}) \cong \pi_{i*}\underline{\Bbbk}_{\widetilde{\mathcal{O}_i}}\langle nm - i^2\rangle$.

Proof. Proposition 4.1 implies that $\pi_{i*}\underline{\Bbbk}_{\widetilde{\mathcal{O}_i}}$ is a shift and twist of $\mathbf{IC}(\overline{\mathcal{O}_i})$, since pushforward by a small resolution sends the constant sheaf to a shift of the intersection cohomology sheaf on the target. The restriction of $\pi_{i*}\underline{\Bbbk}_{\widetilde{\mathcal{O}_i}}\langle nm - i^2\rangle$ to $\mathcal{O}_i$ is isomorphic to $\underline{\Bbbk}_{\mathcal{O}_i}\langle nm - i^2\rangle$, which is our choice of normalization. $\square$

Given a sheaves $\mathcal{F}, \mathcal{G} \in D^b_G(X)$ let us write

$$\mathrm{Hom}^\bullet(\mathcal{F}, \mathcal{G}) := \bigoplus_m \mathrm{Hom}(\mathcal{F}, \mathcal{G}[m]).$$

This is a graded vector space.

Proposition 4.3. *In* $D^b_{P_{m,n} \times P_{n,m}}(G)$ *we have an isomorphism*

$$\mathrm{Hom}^\bullet(\mathbf{IC}(\mathcal{O}_i), \mathbf{IC}(\mathcal{O}_{i'})) \cong \bigoplus_j \mathrm{Hom}^\bullet(f_j^!\mathbf{IC}(\mathcal{O}_i), f_j^*\mathbf{IC}(\mathcal{O}_{i'})).$$

Proof. For flag varieties this is [BGS96, Theorem 3.4.1]. One may reduce to this situation using the quotient equivalence. $\square$

4.2. Our aim in this section is to calculate the weight filtration on the sheaves associated to positive and negative crossings. In order to understand the constituents via the function-sheaf correspondence discussed in Section 2.4, we must calculate the trace of the Frobenius on the stalks of $\mathbf{IC}(\overline{\mathcal{O}_i})$. Base change combined with the Grothendieck-Lefschetz fixed point formula yields

Corollary 4.4. *If $j > i$ and $x \in \mathcal{O}_j(\mathbb{F}_{q^a})$ we have*

$$\operatorname{Tr}(F_{q^a}^*, (\pi_{i*}\underline{\Bbbk}_{\widetilde{\mathcal{O}_i}})_x) = \# \operatorname{Gr}_i^j(\mathbb{F}_{q^a}) = \begin{bmatrix} j \\ i \end{bmatrix}_{q^a}.$$

In the following proposition W denotes the weight filtration:

Proposition 4.5. *One has isomorphisms:*

$$\begin{aligned} \operatorname{gr}_{-i}^W j_! \underline{\Bbbk}_{\mathcal{O}_0}\langle nm\rangle &\cong \mathbf{IC}(\overline{\mathcal{O}_i})(i/2) \\ \operatorname{gr}_i^W j_* \underline{\Bbbk}_{\mathcal{O}_0}\langle nm\rangle &\cong \mathbf{IC}(\overline{\mathcal{O}_i})(-i/2) \end{aligned}$$

Proof. Because taking weight filtrations commutes with forgetting equivariance it is enough to handle the non-equivariant case. Note also that $\mathbf{IC}(\mathcal{O}_i)(i/2)$ is pure of weight $-i$. Thus, by the remarks in Section 2.4, the first statement of the proposition follows the equality of the functions

$$[j_! \underline{\Bbbk}_{\mathcal{O}_0}\langle nm\rangle]_{q^a} = \sum_i [\mathbf{IC}(\mathcal{O}_i)(i/2)]_{q^a}$$

for all $a \geq 1$. Evaluating at a point $x \in \mathcal{O}_j(\mathbb{F}_{q^a})$ we need to verify

$$(-1)^{nm/2}\delta_{0j} q^{-anm/2} = \sum_{0 \leq i \leq j} (-1)^{nm-i^2} q^{a(i^2-nm-i)/2} \begin{bmatrix} j \\ i \end{bmatrix}_{q^a}$$

or equivalently

$$\delta_{0j} = \sum_{0 \leq i \leq j} (-1)^i q^{i(i-1)/2} \begin{bmatrix} j \\ i \end{bmatrix}_q$$

which is a standard identity on q-binomial coefficients. The second statement follows from the first by duality. ☐

Proposition 4.6. *We have equalities*

$$\dim \operatorname{Ext}^1(\mathbf{IC}(\mathcal{O}_i), \mathbf{IC}(\mathcal{O}_{i+1})) = \dim \operatorname{Ext}^1(\mathbf{IC}(\mathcal{O}_{i+1}), \mathbf{IC}(\mathcal{O}_i)) = 1.$$

Proof. By the Verdier self-duality of $\mathbf{IC}$ sheaves, we have an equality of dimensions

$$\dim \operatorname{Ext}^1(\mathbf{IC}(\mathcal{O}_i), \mathbf{IC}(\mathcal{O}_{i+1})) = \dim \operatorname{Ext}^1(\mathbf{IC}(\mathcal{O}_{i+1}), \mathbf{IC}(\mathcal{O}_i)),$$

so we need only give a proof for one.

Using Proposition 4.3, and remembering that

$$\dim \operatorname{Ext}^1(\mathbf{IC}(\mathcal{O}_i), \mathbf{IC}(\mathcal{O}_{i+1})) = \dim \operatorname{Hom}(\mathbf{IC}(\mathcal{O}_i), \mathbf{IC}(\mathcal{O}_{i+1}[1]))$$

one may identify the above space with $H^{2i}(\pi_i^{-1}(x))$ where $x \in \mathcal{O}_{i+1}$. But $\pi_i^{-1}(x) \cong \mathbb{P}^i$ and this space is of dimension 1 as claimed. ☐

Corollary 4.7. *The local chromatographic complex of $j_!\underline{\Bbbk}_{\mathcal{O}_0}\langle nm\rangle$ is the unique complex of the form*

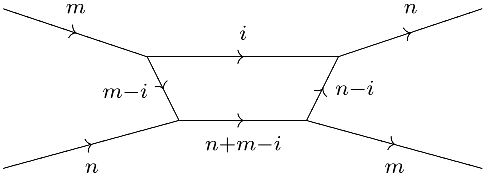
$$0 \to \mathbf{IC}(\mathcal{O}_0) \to \mathbf{IC}(\mathcal{O}_1)\langle 1\rangle \to \cdots \to \mathbf{IC}(\mathcal{O}_i)\langle i\rangle \to \cdots$$

where all differentials are non-zero. Similarly, that for $j_*\underline{\Bbbk}_{\mathcal{O}_0}\langle nm\rangle$, is the unique complex of the form

$$\cdots \to \mathbf{IC}(\mathcal{O}_i)\langle -i\rangle \to \cdots \to \mathbf{IC}(\mathcal{O}_1)\langle -1\rangle \to \mathbf{IC}(\mathcal{O}_0) \to 0$$

also where all differentials are non-zero.

Remark 3. This corollary shows that this chromatographic complex categorifies the MOY expansion of a crossing in terms of trivalent graphs, $\mathbf{IC}(\mathcal{O}_i)$ corresponding to the MOY graph



Proof. The terms in the complex are determined by Proposition 4.5, and Proposition 4.6 implies that the isomorphism type of the complex is just determined by which maps are non-zero. Since $j_!\underline{\Bbbk}_{\mathcal{O}_0}$ and $j_*\underline{\Bbbk}_{\mathcal{O}_0}$ are indecomposible, all these maps must be non-zero. $\square$

5. THE INVARIANT VIA BIMODULES

5.1. The global chromatographic complex of a crossing.

The following lemma gives a description of $\widetilde{\mathcal{O}_i}$ as a "Bott-Samelson" type space:

Lemma 5.1. *We have an isomorphism of $P_{m,n} \times P_{n,m}$-equivariant varieties*

$$\widetilde{\mathcal{O}_i} \cong P_{m,n} \times_{P_{i,m-i,n}} P_{i,m+n-i} \times_{P_{i,n-i,m}} P_{n,m}.$$

Proof. The map sending $[g,h,k]$ to (gV_i, ghV_n, ghk) defines a closed embedding

$$P_{m,n} \times_{P_{i,m-i,n}} P_{i,m+n-i} \times_{P_{i,n-i,m}} P_{n,m} \hookrightarrow \mathrm{Gr}_i^m \times \mathrm{Gr}_n^{n+m} \times G_{m+n}.$$

Its image is given by triples (W,V,g) satisfying $W \subset V$ and $V = gV_n$ which is isomorphic to $\widetilde{\mathcal{O}_i}$ under the map forgetting V. $\square$

Corollary 5.2. *As $S_{m,n} \otimes S_{n,m}$-modules, we have a natural isomorphism*

$$\begin{aligned} H^*_{P_{m,n}\times P_{n,m}}(\widetilde{\mathcal{O}_i}) &\cong M_i \overset{\mathrm{def}}{=} S_{i,m-i,n} \otimes_{S_{i,m+n-i}} S_{i,n-i,m}. \\ \mathbb{H}^*_{P_{m,n}\times P_{n,m}}(\mathbf{IC}(\mathcal{O}_i)) &\cong M_i(nm-i^2) \end{aligned}$$

Proof. The first equality follows immediately from the main theorem of [BL94] (which we restated in the most convenient for our work in our earlier paper [WW08][Theorem 3.3]) and Lemma 5.1. The second is a consequence of Corollary 4.2. $\square$

Now have a global version of Proposition 4.6:

Proposition 5.3. *The spaces of maps $\mathrm{Hom}_{S_{m,n}\otimes S_{n,m}}(M_i(-2i), M_{i-1})$ and $\mathrm{Hom}_{S_{m,n}\otimes S_{n,m}}(M_i(2i), M_{i+1})$ are trivial in degrees < 1 and one dimensional in degree 1.*

Proof. Immediate from [Wil08, Theorem 5.4.1]. In fact, combined with Proposition 4.3, the theorem cited above implies that we have isomorphisms

$$\mathrm{Hom}_{S_{m,n}\otimes S_{n,m}}(M_i(-2i), M_{i-1}) \cong \mathrm{Ext}^\bullet(\mathbf{IC}(\mathcal{O}_i), \mathbf{IC}(\mathcal{O}_{i-1}))$$
$$\mathrm{Hom}_{S_{m,n}\otimes S_{n,m}}(M_i(2i), M_{i+1}) \cong \mathrm{Ext}^\bullet(\mathbf{IC}(\mathcal{O}_i), \mathbf{IC}(\mathcal{O}_{i+1}))$$

with grading degree on module maps matching the homological grading. Thus, this result is equivalent to Proposition 4.6. ☐

Corollary 5.4. *The global chromatographic complex of* $j_!\underline{\Bbbk}_{\mathcal{O}_0}\langle nm\rangle$ *is the unique complex of the form*

$$\mathbf{M}^- = \cdots \xrightarrow{\partial^-_{i+1}} M_{i+1}(nm - i(i+1)) \xrightarrow{\partial^-_i} M_i(nm - i(i-i)) \xrightarrow{\partial^-_{i-1}} \cdots \tag{4}$$

where all differentials are non-zero. Similarly, that for $j_*\underline{\Bbbk}_{\mathcal{O}_0}\langle nm\rangle$*, is the unique complex of the form*

$$\mathbf{M}^+ = \cdots \xrightarrow{\partial^+_{i-1}} M_i(nm - i(1+i)) \xrightarrow{\partial^+_i} M_{i+1}(nm - (i+1)(i+2)) \xrightarrow{\partial^+_{i+1}} \cdots \tag{5}$$

also where all differentials are non-zero.

We note that these are the complexes defined in [MSV, §8], with slight change in grading shift, since they have the same modules, and there is only one such complex up to isomorphism.

We note that these maps have a geometric origin. Consider the correspondence

$$\widetilde{\mathcal{O}_{i+1,i}} = \{(U, W, g) \in \mathrm{Gr}^n_{i+1} \times \mathrm{Gr}^n_i \times G_{n+m} \,|\, gV_n \cap V_m \supset U \supset W\}$$

Obviously, we have natural maps

$$\begin{tikzcd}
& \widetilde{\mathcal{O}_{i+1,i}} \arrow[dl, "p^1_i"'] \arrow[dr, "p^2_i"] & \\
\widetilde{\mathcal{O}_{i+1}} & & \widetilde{\mathcal{O}_i}
\end{tikzcd}$$

Proposition 5.5. *Up to scaling, we have equalities*

$$\partial^-_i = (p^2_i)_*(p^1_i)^* \qquad \partial^+_i = (p^1_i)_*(p^2_i)^*$$

Proof. We note that $(p^2_i)_*(p^1_i)^*$ has the expected degree and is non-zero. Thus it must be ∂^-_i. Similarly with $(p^1_i)_*(p^2_i)$. ☐

5.2. Building the global chromatographic complex I: via canopolis. Now, we are faced with the question of how to build the global chromatographic complex of an arbitrary braid fragment (by which we mean a tangle which can be completed to a closed braid by planar algebra operations).

While the operations we describe are nothing complicated or mysterious, it can be a bit difficult to both be precise and not pile on unnecessary notation. In an effort to give an understandable account for all readers, we give two similar, but slightly different, expositions of how to build the complex for a knot, one quite analogous to Khovanov's exposition in [Kho07] using braids and their closures, and one in the language of planar algebras and canopolises, in the vein of the work of Bar-Natan [BN05] and the first author [Web07].

This approach is based around planar diagrams in sense of planar algebra; a planar diagram is a crossingless tangle diagram in a planar disk with holes. A canopolis is a way of formalizing the process of building up a tangle by gluing smaller tangles into planar diagrams.

Our definition of our geometric invariant can be phrased in this language. Given a tangle T written as a union of smaller tangles T_i in a planar diagram D, the space X_T has a product decomposition $X_T \cong \prod_i X_{T_i}$, and G_T is a subgroup of $\prod_i G_{T_i}$, given by taking the diagonal inside the factors corresponding to the edges on T_i and T_j identified by D.

That is, the sheaf $\mathcal{F}_L$ can be built from the sheaves corresponding to crossings by successive applications of exterior product and restriction of groups. It is easy to understand how each of these affects chromatographic complexes, and our desired invariant can be built piece by piece.

Formally, to each oriented colored tangle diagram in a disk with boundary points $\{p_1, \dots, p_m\}$, we will associate a complex of modules over $R_\Pi = H^*\left(\prod_i BG_{p_i}\right)$, where we use Π to denote all the boundary data of the tangle (the points, their coloring, their orientation).

The association of the category $\mathcal{K}(R_\Pi - \mathsf{mod})$ of complexes up to homotopy over R_Π to the boundary data Π (with their colorings) is a canopolis $\mathcal{K}$, where the functor associated to a planar diagram is an analogue to that used in the canopolis $\mathcal{M}_0$ in [Web07]. The canopolis functor

$$\tilde{\eta} : \mathcal{K}(R_{\Pi_1} - \mathsf{mod}) \times \cdots \times \mathcal{K}(R_{\Pi_k} - \mathsf{mod}) \to \mathcal{K}(R_{\Pi_0} - \mathsf{mod})$$

associated to a planar diagram with outer circle labeled with Π_0 and k inner circles labeled with $\Pi_1, \dots, \Pi_k$ will be given by tensoring with a complex of R_{Π_0}-$R_{\Pi_1} \otimes \cdots \otimes R_{\Pi_k}$ bimodules. We let $R_{\Pi_*} = R_{\Pi_1} \otimes \cdots \otimes R_{\Pi_k}$

Let $\mathcal{A}(\eta)$ be the set of arcs in η, and let α_a, ω_a be the tail and head of $a \in \mathcal{A}(\eta)$, and let n_a be the integer a is colored with. Associated to each arc, we associate the sequence

$$(e_1(\omega_a) - e_1(\alpha_a), \dots, e_{n_a}(\omega_a) - e_{n_a}(\alpha_a)),$$

which identifies the classes $e_i \in H^*(BG_n)$ corresponding to the elementary symmetric polynomials (geometrically, these are the Chern classes of the tautological bundle on BG_n) for the endpoints connected by the arc. To our diagram, we associate the concatenation of these sequences.

Let $\kappa(\eta)$ be the Koszul complex over $R_{\Pi_0} \otimes \cdots \otimes R_{\Pi_k}$ of this concatenated sequence for our diagram η, which we think of as a bimodule with the R_{Π_0}-action on the left and the R_{Π_*} on the right.

Definition 5.6. *The canopolis functor $\tilde{\eta}$ associated to the diagram η is $\kappa(\eta) \otimes_{R_{\Pi_*}} -$.*

Proposition 5.7. *The map sending a tangle T to the global chromatographic complex of $\mathcal{F}_T$ is a canopolis map.*

Proof. We simply need to justify why tensoring with such a Koszul resolution (which is a free resolution of the diagonal bimodule for $H^*(BG_{p_i})$) is the same as changing G_T to only include the diagonal subgroup of $G_{\omega_a} \times G_{\alpha_a}$. This is one of the basic results of [BL94] (as we mentioned earlier, this is rephrased most conveniently for us in [WW08, Theorem 3.3]). □

Remark 4. We note that this construction at no point used the fact that our diagram should be a braid fragment; unfortunately, it is unclear whether our construction will be invariant under the oppositely oriented Reidemeister II move, as with Khovanov-Rozansky's original construction (see, for example, [Web07, §3]) though we will note that proving invariance under this move for the all 1's labeling is sufficient to imply it for all labeling, by the same cabling arguments we will use later.

5.3. Building the global chromatographic complex II: via bimodules. A less flexible, but perhaps more familiar, perspective is to associate to each braid a complex of bimodules, in a manner similar to [Kho07] (though the same complex had previously appeared in other works on geometric representation theory). In the case where all labels are 1, our construction will coincide with Khovanov's.

As in Section 3.3, we let β be a braid with n strands, and $\mathbf{n} = (i_1, \dots, i_m)$ be the labels of the top end of the strands (so $\mathbf{n}\beta$ is the labeling of the bottom end). In that section, we showed the our invariant can also be described in terms of the chromatographic complex of a sheaf Φ_β on G_N.

This sheaf has the advantage that it can be built from the sheaves for smaller braids by convolution of sheaves. However, convolution of sheaves is a geometric operation which is not always easy to understand. Thus, we will give a description of it using tensor product of bimodules. Let $F(\beta)$ be the $P_{\mathbf{n}} \times P_{\mathbf{n}\beta}$-equivariant global chromatographic complex of Φ_β, considered as a complex of bimodules over $H^*(BP_{\mathbf{n}})$ and $H^*(BP_{\mathbf{n}\beta})$.

Proposition 5.8. *We have natural isomorphisms*

$$F(\beta\beta') \cong F(\beta) \otimes_{H^*(BP_{\mathbf{n}\beta})} F(\beta').$$

Proof. Consider the exterior product $\Phi_\beta \boxtimes \Phi_{\beta'}$ on $G_N \times G_N$. The $P_{\mathbf{n}} \times P_{\mathbf{n}\beta} \times P_{\mathbf{n}\beta} \times P_{\mathbf{n}\beta\beta'}$-equivariant chromatographic complex of this is $F(\beta) \otimes_{\mathbb{C}} F(\beta')$. If we restrict to the diagonal $P_{\mathbf{n}\beta}$, then this complex is $F(\beta) \overset{L}{\otimes}_{H^*(BP_{\mathbf{n}\beta})} F(\beta')$. By the equivariant formality of all simple, Schubert-smooth perverse sheaves on a partial flag variety, $F(\beta)$ is free as a right module, so it is not necessary to take derived tensor product.

By the convolution description, we have

$$\Phi_{\beta'\beta} \cong {}^{P_{\mathbf{n}} \times P_{\mathbf{n}\beta\beta'}}_{P_{\mathbf{n}} \times P_{\mathbf{n}\beta} \times P_{\mathbf{n}\beta\beta'}} \mu_*(\Phi_{\beta,\beta'})$$

where $\mu : G_N \times G_N \to G_N$. Since $G/P_{\mathbf{n}\beta}$ is projective, this map simply has the effect of forgetting the $H^*(BP_{\mathbf{n}\beta})$ action on each page of the chromatographic spectral sequence. $\square$

Thus, we can construct $F(\beta)$ just by knowing the complex $F(\sigma_i^{\pm 1})$ for the elementary twists $\sigma_i^{\pm 1}$. However, first we must compute the corresponding sheaves. Given $\mathbf{n}$, we let $Q_j = P_{i_1, \dots, i_j + i_{j+1}, \dots, i_n}$, and let $\mathring{Q}_j = Q_j - Q_0$.

Proposition 5.9. *We have isomorphisms*

$$\Phi_{\sigma_i} = j_* \underline{\Bbbk}_{\mathring{Q}_i} \langle i_i i_{i+1} \rangle \qquad \Phi_{\sigma_i^{-1}} = j_! \underline{\Bbbk}_{\mathring{Q}_i} \langle i_i i_{i+1} \rangle,$$

where $j : \mathring{Q}_i \hookrightarrow G_N$ is the obvious inclusion.

The global complex of this is very close to the complex M^+ described in (4), considered as a complex of $R_{i_i, i_{i+1}}$-R_{i_{i+1}, i_i} bimodules. However, we must extend scalars to get a complex of $R_{\mathbf{n}}$-$R_{\sigma_i \mathbf{n}}$ bimodules

Proposition 5.10. $F(\sigma_i^{\pm 1}) = R_{i_1,\dots,i_{i-1}} \otimes_{\mathbb{Q}} \mathbf{M}^{\pm} \otimes_{\mathbb{Q}} R_{i_{i+2},\dots,i_k}$.

Again, this is precisely the complex given in [MSV, §8] up to grading shift.

If $\mathbf{n}\beta = \mathbf{n}$, then we can close this braid to a link. Our definition of the knot invariant for this link is the equivariant chromatographic complex for the diagonal $P_{\mathbf{n}}$-action. By the authors' previous work [WW08, Theorem 1.2], this coincides with the Hochschild homology $HH^*(F(\beta))$, applied termwise of the complex $F(\beta)$.

Proposition 5.11. *The cohomology of the complex $HH^*_{R_{\mathbf{n}}}(F(\beta))$ coincides with the invariant $\mathcal{A}_2(\hat{\beta})$ of the closure of the braid.*

In fact, the chromatographic spectral sequence is exactly the natural spectral sequence

$$\mathcal{H}^i\big(HH^j(F(\beta))\big) \Rightarrow \mathcal{H}^{i+j}(R_{\mathbf{n}} \overset{L}{\otimes}_{R_{\mathbf{n}} \otimes R_{\mathbf{n}}} F(\beta)).$$

Proof. Let $\pi : G_N \to pt$, and consider the object $\pi_*\Phi_\beta$ in the equivariant derived category $D_{P_{\mathbf{n}} \times P_{\mathbf{n}}}(pt)$. Under the equivalence to $R_{\mathbf{n}}$-dg-bimodules given in [WWa, Theorem 7], this is sent to the complex $F(\sigma)$. Similarly, the weight filtration is sent to that induced by thinking of $F(\beta)$ as a complex. Thus, the spectral sequences match under this equivalence. □

Since $\mathcal{H}^*(HH^*(F(\beta)))$ is precisely the invariant proposed by [MSV], Theorem 1.4 follows immediately.

6. DECATEGORIFICATION

We also wish to show that our knot invariant is, in fact, a categorification of the HOMFLYPT polynomial.

6.1. A categorification of the Hecke algebra.

This requires a few basic results about the relationship between sheaves on G_n and the Hecke algebra $\mathbf{H}_n$. As usual, $B = P_{1,\dots,1}$ is the standard Borel.

Definition 6.1. *The Hecke algebra $\mathbf{H}_n$ is the algebra over $\mathbb{Z}[q^{1/2}, q^{-1/2}]$ given by the quotient of the group algebra of the braid group $\mathcal{B}_n$ by the quadratic relation*

$$(\sigma_i + q^{1/2})(\sigma_i - q^{-1/2}) = 0$$

for each elementary twist σ_i.

Proposition 6.2 ([KW01])**.** *The Grothendieck group of the equivariant derived category $D^b_{B\times B}(G_n)$ is isomorphic to the Hecke algebra $\mathbf{H}_n$, with the convolution product decategorifying to the algebra product in $\mathbf{H}_n$.*

This map is fixed by the assignment

$$[j_*\underline{\Bbbk}_{Bs_iB}] \mapsto q^{1/2}\sigma_i$$

where $j : Bs_iB \hookrightarrow G_n$ is the obvious inclusion.

Let $\mathcal{F}$ be a $B \times B$-equivariant sheaf on G_n. Then we have a map

$$\mathcal{E}_B(G;\mathcal{F}) = \sum_{i,j,k} (-1)^{\ell} q^{j/2} t^k \dim \mathbb{H}_{B_\Delta}^{j-\ell;\frac{j-k}{2}}(\mathrm{gr}_\ell^W \mathcal{F})$$

sending the class of $\mathcal{F}$ in the Grothendieck group to the bi-graded Euler characteristic of its global chromatographic complex.

This map agrees with a previously known trace on the Hecke algebra, a fact that the authors have proven in a separate note, due to its independent interest and separate connection to the question of constructing Markov traces on general Hecke algebras.

Proposition 6.3. [WWb, Theorem 1] *The map* $\mathcal{E}_B(G_n; -)$ *is the Jones-Ocneanu trace* Tr [Jon87] *on* $\mathbf{H}_n$ *with appropriate normalization factors.*

Remark 5. This geometric definition applies equally well to any simple Lie group, and defines a canonical trace on the Hecke algebra for any type. In fact, our construction can be modified in a straightforward way to a "triply graded homology" invariant on all Artin braid groups. In type B, this can be interepreted as a homological knot invariant for knots in the complement of a torus.

6.2. Decategorification for colored HOMFLYPT. To apply this result, we must relate our construction to the categorification of the Hecke algebra above. Recall that if σ is a braid labeled all with 1's, then Φ_σ is an object of $D^b_{B\times B}(G_n)$

Proposition 6.4. *The class* $[\Phi_\sigma] \in \mathbf{H}_n$ *is the image of* σ *under the natural map* $\mathcal{B}_n \to \mathbf{H}_n$*.*

This, combined with Proposition 6.3, gives a new proof of the result of Khovanov [Kho07] that all components are labeled with 1, the invariant

$$\mathcal{E}(L) = \mathcal{E}_{G_L}(X_L; \mathcal{F}_L) = \sum_{i,j,k} (-1)^\ell q^j t^k \dim \mathcal{A}_2^{j;k;\ell}(L)$$

is the appropriately normalized HOMFLYPT polynomial of L. We wish to extend this to the colored case. For this, we must use a "cabling/projection" formula.

Consider a closable colored braid σ, and let $P = P_{\mathbf{n}}$ and $G = G_N$. We have defined a $P \times P$-equivariant sheaf Φ_σ on G by the multiplication map $m : X_\sigma \to G$.

Theorem 6.5. *For any colored link L, the Euler characteristic* $\mathcal{E}(L)$ *is the (suitably normalized) colored HOMFLY polynomial.*

In order to prepare for the proof, we show a pair of lemmata. Let σ_{cab} denote the cabling of σ in the blackboard framing with multiplicities given by the colorings, thought of as colored with all 1's.

Lemma 6.6. *We have an isomorphism of* $P \times B$*-equivariant sheaves*

$$\mathrm{res}^{P\times P}_{P\times B}\Phi_\sigma \cong \mathrm{ind}^{P\times B}_{B\times B}\Phi_{\sigma_{cab}}.$$

Proof. The proof is a straightforward induction on the length of σ; left to the reader. □

Let $\lambda_{\mathbf{n}}$ be the partition given by arranging the parts of $\mathbf{n}$ in decreasing order, and let $\lambda^t_{\mathbf{n}}$ be its transpose. Let $\pi_{\mathbf{n}}$ be the projection in the Hecke algebra to the representations indexed by Young diagrams less than $\lambda^t_{\mathbf{n}}$ in dominance order. Alternatively, if we identify $\mathbf{H}_N$ with the endomorphisms of $V^{\otimes N}$ where V is the standard representation of $U_q(\mathfrak{sl}_m)$ for $m \geq n$, then this is the projection to $\wedge^{i_1} V \otimes \cdots \otimes \wedge^{i_n} V$.

Lemma 6.7. *We have* $[\mathrm{res}^{B\times B}_{P\times B}\mathrm{ind}^{P\times B}_{B\times B}\Phi] = q_P \pi_P[\Phi]$.

Proof. First consider the case where $P = G$. In this case, the sheaf $\mathrm{res}^{B\times B}_{G\times B}\mathrm{ind}^{G\times B}_{B\times B}\Phi$ has a filtration whose successive quotients are of the form $\mathbb{H}^i(\Phi) \otimes \underline{\Bbbk}_G$. Thus we have

$$[\mathrm{res}^{B\times B}_{G\times B}\mathrm{ind}^{G\times B}_{B\times B}\Phi] = \dim_q \mathbb{H}^*(\Phi) \cdot [\underline{\Bbbk}_G].$$

It is a classical fact that $[\underline{\Bbbk}_G] = q_G\pi_G$; here π_G is just the projection to $\wedge^N V$. This computation immediately extends to the general case. □

Remark 6. This proposition shows why our approach works for colored HOMFLYPT polynomials, but would need to be modified to approach the HOMFLY polynomials for more general type A representations; we lack a good categorification of most of the projections in the Hecke algebra, but π_P has a beautiful geometric counterpart. This may be related to the fact that π_P is the projection not just to a subrepresentation, but in fact to a cellular ideal in $\mathbf{H}_n$.

Proof of Theorem 6.5. Immediately from Lemmata 6.6 and 6.7, we have $[\mathrm{res}^{P\times P}_{B\times B}\Phi_\sigma] = q_P\pi_P[\Phi_{\sigma_{cab}}]$. Thus

$$\begin{aligned}\mathcal{E}_P(G;\Phi_\sigma) &= q_P^{-1}\mathcal{E}_B(G;\mathrm{res}^{P\times P}_{B\times B}\Phi_\sigma)\\ &= \mathrm{Tr}(q_P^{-1}[\mathrm{res}^{P\times P}_{B\times B}\Phi_\sigma])\\ &= \mathrm{Tr}(\pi_P[\Phi_{\sigma_{cab}}])\end{aligned}$$

By the "projection/cabling" formula (see, for example, [LZ, Lemma 3.3]), this is precisely the colored HOMFLY polynomial. □

7. THE PROOF OF INVARIANCE: $\mathrm{GL}(2)$

We first concentrate on the simpler case of $\mathrm{GL}(2)$ before attacking the general case. In this case, we will obtain an invariant which matches the HOMFLYPT homology of Khovanov-Rozansky [KR08, Kho07], so the section below can be thought of as a geometric proof of the invariance of this homology theory.

Recall that if σ is a braidlike diagram on n strands we described in Section 3.3 a map

$$m : X_\sigma \to G_n$$

equivariant with respect to $\phi : G_\sigma \to T\times T$, where $T\times T$ acts on G_n by left and right multiplication. This map gives rise to a functor

$$^{T\times T}_{G_\sigma} m_* : D^+_{G_\Gamma}(X_\Gamma) \to D^+_{T\times T}(G_n)$$

and we denoted the image of $\mathcal{F}_\sigma$ by Φ_σ. We saw that this functor preserves weight filtrations.

Now suppose that w is an element of the symmetric group on n-letters (which we regard as permutation matrices in G_n) and that $\sigma = \sigma_{i_1}\sigma_{i_2}\dots\sigma_{i_p}$ is a (positive) braid in the standard generators corresponding to a reduced expression $s_{i_1}\dots s_{i_p}$ for w.

It is straightforward to see that if we restrict m to the open set $\tilde{U}$ in G_Γ consisting of tuples $(g_1,\dots,g_p)$ where each $g_i\in U$ (where U denotes the open Bruhat cell in G_2) then we may factor m as

$$\tilde{U} \to \tilde{U}/\ker\phi \to G_n \tag{6}$$

where the first map is a quotient by a free action, and the second map is an isomorphism.

Moreover, if we denote by B the subgroup of upper triangular matrices, then the image of the restriction of m to $\tilde{U}$ is contained in Schubert cell BwB. As B/T is acyclic it follows that

$$\mathrm{ind}^{B\times B}_{T\times T}(^{T\times T}_{G_\Gamma} m_*\mathcal{F}_\sigma) = j_{w!}\underline{\Bbbk}_{BwB}\langle\ell(w)\rangle.$$

(Here j_w denotes the inclusion of the Bruhat cell BwB into G_n). It is also follows from the above considerations that $\mathrm{ind}_{T\times T}^{B\times B}({}^{T\times T}_{G_\Gamma} m_* -)$ preserves weight filtrations. As we are only concerned with the spectral sequence computing $\mathcal{F}_\sigma$ which is unchanged by induction we can (and will) replace $\mathcal{F}_\sigma$ by $j_{w!}\underline{\Bbbk}_{BwB}\langle \ell(w)\rangle$ in the above situation.

Proposition 7.1. *Theorem 1.2 holds in the case where all strands are labeled by 1.*

Proof. As usual with proofs that knot invariants defined in terms of a projection are really invariants, we check that our description is unchanged by the Reidemeister moves. Since we only consider closed braids, we only need to check Reidemeister II and III in the braid-like case, when all strands are coherently oriented. Those who prefer to use the Markov theorem can consider the proof of Reidemeister I as a proof of the Markov 1 move, and the Reidemeister II and III calculations as proving the independence of the presentation of our braid in terms of elementary twists *and* of the Markov 2 move (which only uses Reidemeister IIa).

In each case, we will use the fact that while we wish to compare the pushforwards of sheaves corresponding to diagrams L and L' on from X_L/G_L and $X_{L'}/G_{L'}$ to a point, we can accomplish this by showing that their pushforwards by any pair of maps to any common common space coincide. Being able to use these sort of techniques is one of the principal advantages of a geometric definition over a purely algebraic one.

Reidemeister I: Consider the following tangles:

(7) $D =$  $D' =$  .

To simplify notation we denote the associated varieties X, X' and groups G, G' respectively. We have $X = G_2$ and $X' = G_1$, $G = G_1^3$ and $G' = G_1^2$. The determinant gives a map

$$d : X \to X'$$

which is equivariant with respect to the map $\phi : G \to G'$ forgetting the factor corresponding to the internal edge. Reidemeister I will result from an isomorphism

$${}^{G'}_{G} d_* \mathcal{F}_D \cong \mathcal{F}_{D'}$$

compatible with the weight filtrations on both sheaves. Note that the weight filtration on $\mathcal{F}_{D'}$ is trivial, whereas that on $\mathcal{F}_D$ is not.

Let $B \overset{a}{\hookrightarrow} X \overset{b}{\hookleftarrow} BsB$ be the decomposition of $X = G_2$ into its two Bruhat cells. We have an distinguished triangle

$$a_!a^!\underline{\Bbbk}_X\langle 1\rangle \to \underline{\Bbbk}_X\langle 1\rangle \to b_*b^*\underline{\Bbbk}_X\langle 1\rangle \xrightarrow{[1]}$$

turning the triangle gives the weight filtration on $b_*\underline{\Bbbk}_{BsB}\langle 1\rangle$:

$$\underline{\Bbbk}_X\langle 1\rangle \to b_*\underline{\Bbbk}_{BsB}\langle 1\rangle \to a_*\underline{\Bbbk}_B(-{}^1\!/_2) \xrightarrow{[1]} . \tag{8}$$

In the following we analyze the effect of ${}^{G'}_{G} d_*$ on this triangle.

The restriction of d to $BsB \subset X$ is a trivial $G_1 \times \mathbb{A}^2$-bundle over X'. One may easily check that $\ker \phi$ acts freely on the multiplicative group in the fiber. It follows that

$$ {}^{G'}_{G} d_* b_* \underline{\Bbbk}_{BsB} \cong \underline{\Bbbk}_{X'}. $$

On the other hand, the restriction of d to $B \subset X$ yields a trivial $G_1 \times \mathbb{A}^1$ bundle, with $\ker \phi$ only acting on $\mathbb{A}^1$. It follows that

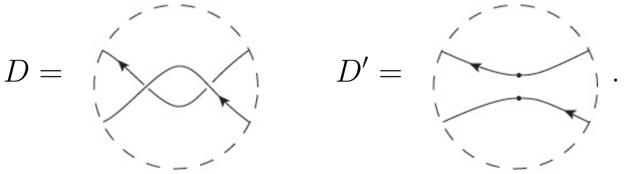
$$ {}^{G'}_{G} d_* a_* \underline{\Bbbk}_{B} = H^\bullet(\mathbb{P}^\infty) \otimes H^\bullet(G_1) \otimes \underline{\Bbbk}_{X'}. $$

Applying ${}^{G'}_{G} d_*$ to (8) and using the above isomorphisms we obtain

$$ {}^{G'}_{G} d_* \underline{\Bbbk}_{X} \langle 1 \rangle \to \underline{\Bbbk}_{X'} \langle 1 \rangle \to H^\bullet(\mathbb{P}^\infty) \otimes H^\bullet(G_1) \otimes \underline{\Bbbk}_{X'} \langle -1/2 \rangle \xrightarrow{[1]} . $$

As $\mathrm{Hom}(\underline{\Bbbk}_{X'}, \underline{\Bbbk}_{X'}[i]) = H^i_{G'}(X')$ is zero for $i < 0$ we conclude that the second arrow above is zero. Hence the filtration on $\underline{\Bbbk}_{X'}$ may be taken to be trivial (and therefore agrees with that on $\mathcal{F}_{D'}$ up to $\langle 1 \rangle$).

Reidemeister IIa: Here we are concerned with the two tangles:



We denote the associated varieties and groups X, X', G, G'. We denote by m the multiplication map $X \to G_2$ considered at the start of this section. We regard X' as the diagonal matrices inside G_2.

We have seen that ${}^{G'}_{G} m_*$ preserves weight filtrations, and hence we may ignore weight filtrations when comparing ${}^{G'}_{G} m_* \mathcal{F}_D$ and $\mathcal{F}_{D'}$. The map $B \to X'$ forgetting the off-diagonal entry is acyclic, and therefore it is enough to show that ${}^{G'}_{G} m_* \mathcal{F}_D \cong \underline{\Bbbk}_B$.

We decompose G_2 into its Bruhat cells $B \overset{a}{\hookrightarrow} G_2 \overset{b}{\hookleftarrow} BsB$ as before. We claim we have isomorphisms:

$$ {}^{G'}_{G} m_* (a_* \underline{\Bbbk}_B \boxtimes b_! \underline{\Bbbk}_{BsB}) \cong b_! \underline{\Bbbk}_{BsB} \tag{9} $$

$$ {}^{G'}_{G} m_* (\underline{\Bbbk}_G \boxtimes a_* \underline{\Bbbk}_B) \cong \underline{\Bbbk}_G \tag{10} $$

$$ {}^{G'}_{G} m_* (\underline{\Bbbk}_G \boxtimes \underline{\Bbbk}_G) \cong \underline{\Bbbk}_G \oplus \underline{\Bbbk}_G \langle -2 \rangle \tag{11} $$

$$ {}^{G'}_{G} m_* (\underline{\Bbbk}_G \boxtimes b_! \underline{\Bbbk}_{BsB}) \cong \underline{\Bbbk}_G \langle -2 \rangle \tag{12} $$

(As always we regard the exterior tensor product of equivariant sheaves on G_2 as an equivariant sheaf on X via restriction.)

Indeed, (9) and (10) follow from the fact that the restriction of m to $B \times G$ or $G \times B$ is a trivial B-bundle, with $\ker \phi$ acting freely on the multiplicative groups in the fiber. The factorization (6) of m as "essentially a $\mathbb{P}^1$-bundle" implies (11). Then (12) follows from the others by taking the exterior tensor product of $\underline{\Bbbk}_G$ with the distinguished triangle $b_! \underline{\Bbbk}_{BsB} \to \underline{\Bbbk}_G \to a_* \underline{\Bbbk}_B \to$ and applying ${}^{G'}_{G} m_*$.

Now B is smooth of codimension 1 inside G_2 and $a^! \underline{\Bbbk}_G = \underline{\Bbbk}_B \langle -2 \rangle$ and we have an exact triangle

$$ a_* \underline{\Bbbk}_B \langle -2 \rangle \to \underline{\Bbbk}_G \to b_* \underline{\Bbbk}_{BsB} \xrightarrow{[1]} . $$

Taking the exterior tensor product with $b_! \underline{\Bbbk}_{BsB}$, applying ${}^{G'}_{G}m_*$ and using the above isomorphisms we obtain a distinguished triangle

$$b_! \underline{\Bbbk}_{BsB}\langle -2\rangle \to \underline{\Bbbk}_G\langle -2\rangle \to {}^{G'}_{G}m_*(b_*\underline{\Bbbk}_{BsB} \boxtimes b_!\underline{\Bbbk}_{BsB}) \xrightarrow{[1]} \tag{13}$$

Note that $\mathrm{Hom}(b_!\underline{\Bbbk}_{BsB}, \underline{\Bbbk}_G)$ is one dimensional and contains the adjunction morphism $b_!b^!\underline{\Bbbk}_G \to \underline{\Bbbk}_G$. By considering its dual, one may show that the first arrow in (13) is non-zero. It follows that this arrow is the adjunction morphism (up to a non-zero scalar) and we have an isomorphism:

$${}^{G'}_{G}m_*(b_*\underline{\Bbbk}_{BsB} \boxtimes b_!\underline{\Bbbk}_{BsB}) \cong \underline{\Bbbk}_B\langle -2\rangle$$

Finally note that by definition $\mathcal{F}_D$ is $b_*\underline{\Bbbk}_{BsB} \boxtimes b_!\underline{\Bbbk}_{BsB}\langle 2\rangle$ and so

$${}^{G'}_{G}m_*\mathcal{F}_D \cong \underline{\Bbbk}_B$$

which finishes the proof of invariance under Reidemeister II.

Reidemeister III: This follows immediately from the considerations at the beginning of this section. Indeed, if σ and σ' are the diagrams corresponding to the words $\sigma_1\sigma_2\sigma_1$ and $\sigma_2\sigma_1\sigma_2$ we have maps

$$X_\sigma \xrightarrow{m} G_3 \xleftarrow{m'} X_{\sigma'}$$

and

$${}^{T\times T}_{G_\sigma}m_*\mathcal{F}_\sigma \cong j_{w_0}\underline{\Bbbk}_{Bw_0B} \cong {}^{T\times T}_{G_\sigma}m'_*\mathcal{F}_{\sigma'}$$

(here w_0 indicates the longest element in S_3). $\square$

8. THE PROOF OF INVARIANCE: $\mathrm{GL}(n)$

Now, we expand to the full case of all possible positive integer labels.

Proof of Theorem 1.2. All of the Reidemeister moves can simply be reduces to the corresponding statement for the cabling with the all 1's labeling. Interestingly, the same trick was used in [MSV] to prove invariance in a special case. Almost certainly our proof could be rephrased in a purely algebraic language like their paper, though at the moment it is unclear how.

Reidemeister IIa & III: Here we need only establish the isomorphisms of $P\times P$-equivariant sheaves

$$\Phi_{\sigma_i} \star \Phi_{\sigma_i^{-1}} \cong \underline{\Bbbk}_P \qquad \Phi_{\sigma_i} \star \Phi_{\sigma_{i+1}} \star \Phi_{\sigma_i} \cong \Phi_{\sigma_{i+1}} \star \Phi_{\sigma_i} \star \Phi_{\sigma_{i+1}}$$

Lemma 6.6 implies that these hold as $P \times B$ equivariant sheaves, applying the invariance for the all 1's labeling to the cable.

In fact, both are the $*$-inclusion of a local system on a $P \times P$-orbit: P itself in first case, the $P \times P$ orbit of the permutation corresponding to the cabling of $\sigma_i\sigma_{i+1}\sigma_i$. Since the stabilizer of any point under $P \times P$ is connected, any $P \times B$ equivariant local system on an orbit has at most one $P \times P$ equivariant structure, and this equality holds as $P \times P$ equivariant sheaves.

Reidemeister I: We again use the "cabling/projection" philosophy, but this argument requires a bit more subtlety. We are interested in the chromatographic complex of a single crossing with its right ends capped off, that is, the tangle projection denoted by D in (7). To construct the sheaf $\mathcal{F}_D$, we take $U \subset G_{2n}$, as defined in (3), and consider $j_*\underline{\Bbbk}_U$ or $j_!\underline{\Bbbk}_U$, depending on whether our crossing is positive or negative. These cases are Verdier dual, and the proofs of invariance are essentially identical, so we will treat the positive case, and only note where the negative differs.

We consider the action on G_{2n} of $G_{n,n}$ on the left *and* the right. By convention, we let G^1_n denote the first copy of $G_n \subset G_{n,n}$ and G^2_n the second. As before, we let T_n be diagonal matrices in G_n, and we use T^1_n, T^2_n for the inclusions into the two factors. We let $G^{1,1,2}_{n,n,n}$ denote $G^1_n \times G^1_n \times (G^2_n)_\Delta$, that is, the left and right action of G^1_n, and the conjugation action of G^2_n.

In order to prove the theorem, what we must do is consider the $G^{1,1,2}_{n,n,n}$-equivariant global chromatographic complex of $\mathcal{F}_D$ as a $H^*(BG^1_n)$-bimodule, and show that it matches that of an untwisted strand (the diagram denoted D' in (7)).

Note that for any G_n sheaf $\mathcal{F}$ on any G_n-space X, the inclusion of the symmetric group as permutation matrices normalizing T_n gives an action of S_n on $\mathbb{H}^*_{T_n}(X; \mathrm{res}^{G_n}_{T_n}\mathcal{F})$.

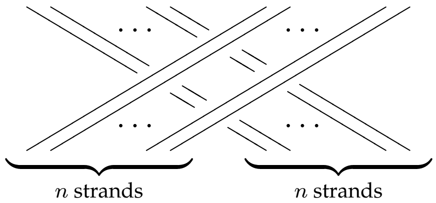
Lemma 8.1. *The natural transformation of functors*

$$\mathbb{H}^*_{G^{1,1,2}_{n,n,n}}(G_{2n}; -) \to \mathbb{H}^*_{G^{1,1}_{n,n}\times T^2_n}(G_{2n}; \mathrm{res}^{G^{1,1,2}_{n,n,n}}_{G^{1,1}_{n,n}\times T^2_n} -)$$

is the inclusion of the S_n-invariants for the permutation action on T^2_n.

Proof. This is the abelianization theorem for equivariant cohomology. □

Let $\hat{U}$ be the Bruhat cell $Bw^{n,n}_{2n}B$ where $w^{n,n}_{2n}$ is the permutation which switches i and $i \pm n$, and let $\hat{j}$ be its inclusion to G_{2n}. We note that $\hat{j}_*\underline{\Bbbk}_{\hat{U}}$ is Φ_σ where σ is the braid given by the n-cabling of a single crossing:



Lemma 8.2. *The $G^{1,1}_{n,n} \times T^2_n$-equivariant global chromatographic complex of $j_*\underline{\Bbbk}_U$ is isomorphic to the $T^{1,1}_{n,n} \times T^2_n$-equivariant for $\hat{j}_*\underline{\Bbbk}_{\hat{U}}$, with the bimodule structure restricted to $H^*(BG^{1,1}_{n,n}) \subset H^*(BT^{1,1}_{n,n})$.*

Proof. Let $Q = G^1_n \cap B$ be the upper-triangular matrices in G_n, given the natural embedding in $G_{n,n}$. Then

$$\mathrm{ind}^{G^{1,1}_{n,n}\times T^2_n}_{T^{1,1}_{n,n}\times T^2_n} j_*\underline{\Bbbk}_{\hat{U}} \cong \mathrm{ind}^{G^{1,1}_{n,n}\times T^2_n}_{Q\times Q\times T^2_n}\mathrm{ind}^{Q\times Q\times T^2_n}_{T^{1,1}_{n,n}\times T^2_n} j_*\underline{\Bbbk}_{\hat{U}} \cong \mathrm{res}^{G^{1,1,2}_{n,n,n}}_{T^{1,1}_{n,n}\times T^2_n} j_*\underline{\Bbbk}_U$$

The first induction leaves chromatographic complexes unchanged, which Q and T^1_n are homotopy equivalent, and $j_*\underline{\Bbbk}_{\hat{U}}$ is smooth on $Q \times Q$-orbits.

For the second, we have a projective map

$$\mu : G_n \times_Q \overline{\hat{U}} \times_Q G_n \to G_{2n}$$

which induces an isomorphism

$$G_n \times_Q \hat{U} \times_Q G_n \cong U.$$

By [WWa, Theorem 5], under taking equivariant cohomology, induction of sheaves corresponds to the restriction of scalars, and since G_n/Q is projective this result extends to all terms in the chromatographic spectral sequence. □

Of course, by definition, the $T^{1,1}_{n,n} \times T^2_n$-equivariant chromatographic complex for $\hat{j}_*\underline{\Bbbk}_{\hat{U}}$ is just the complex of bimodules for the tangle diagram D_{cab} corresponding to closing the right half of the strands in the braid above. Applying the invariance result for labelings all with 1's, this is the same as the complex corresponding to a full twist of n strands.

Note that if we consider a negative crossing, we will have to include n times the usual shift for removing a negative stabilization, but this is easily accounted for in the normalization.

Of course, restricted to symmetric polynomials (that is, $H^*(BG_n)$), every Soergel bimodule is a number of copies of the regular bimodule, and every map in the complex for a single crossing splits, so restricted to $H^*(BG_n)$, the complex attached to a braid labeled all with 1's is homotopic to a single copy of $H^*(BT_n)$ with the regular bimodule action and standard S_n-action. By Lemma 8.1, to obtain the $G^{1,1,2}_{n,n,n}$-equivariant global chromatographic complex we simply take S_n-invariants and thus we obtain a single copy of the regular bimodule for $H^*(BG_n)$, which is exactly what we desired. ☐

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