Representation theory and a strange duality for symplectic varieties

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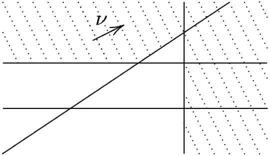
Outline

Hyperplane arrangements

Let's start with a little notation.

A **polarized arrangement** $\mathcal{V} = (V, \xi, \nu)$ is

1. A subspace $V \subset \mathbb{R}^n$.
2. An element $\xi \in \mathbb{R}^n/V$ (a coset $V + \xi \subset \mathbb{R}^n$).
3. An element $\nu \in V^*$ (a direction in V).



We'll always assume that this choice is generic.

This picture above is of $V + \xi$. The hyperplanes in the arrangement are the vanishing sets of $t_i|_{V+\xi}$ (where the t_i are the coordinates on $\mathbb{R}^n$).

The **chambers** of $\mathcal{V}$ are the connected components of $(V + \xi) \cap (\mathbb{R}^\times)^n$.

We call a chamber **bounded** if ν achieves a maximum on it. We let $\mathcal{B}$ denote the set of bounded chambers.

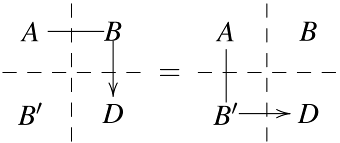
A mysterious algebra

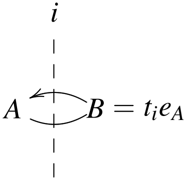
From such an arrangement, one can build an algebra $A(\mathcal{V})$ over $\mathrm{Sym}^\bullet(V^*)$, generated by elements

- c_{AB} for all chambers A, B which are adjacent across a hyperplane.
- idempotents e_A for all chambers A.
- the coordinate functions t_i on $\mathbb{R}^n$, pulled back to V.

with the relations

- $c_{AB}e_{B'} = c_{AB}\delta_B^{B'}$ and $e_{A'}c_{AB} = c_{AB}\delta_A^{A'}$.
- $c_{AB}c_{BD} = c_{AB'}c_{B'D}$.
- $e_A = 0$ if A is not bounded.
- $c_{AB}c_{BA} = t_i e_A$.





Good properties

Despite its mysterious origins, this algebra is quite well behaved.

Definition

An algebra A **quasi-hereditary** *if it has an exceptional collection of* **standard modules** *which generate $A-\mathsf{mod}$ (like Verma modules in category $\mathcal{O}$).*

A positively graded algebra $A = A_0 \oplus A_{>0}$ **Koszul** *if the two natural gradings on $A^\star = \mathrm{Ext}^*_A(A_0, A_0)$ agree. $A^\star$ is called the* **Koszul dual** *of A.*

There's an equivalence of derived categories $D(A-\mathsf{gmod}) \cong D(A^\star-\mathsf{gmod})$.

Theorem (BLPW)

- *$A(\mathcal{V})$ is quasi-hereditary.*
- *$A(\mathcal{V})$ is Koszul.*
- *The center $Z(A(\mathcal{V}))$ is the reduced Stanley-Reisner ring of $\mathbb{R}^n \to V^*$.*

Examples

A few of these are algebras you might have heard of before:

- If
$$V = \{\mathbf{x} \in \mathbb{R}^n | \sum_i x_i = 0\},$$
then the hyperplane arrangement is the faces of a $n-1$-simplex, and the associated category is the block of category $\mathcal{O}$ for $\mathfrak{sl}_n$ including the simple $L_{m\omega_1-\rho}$ (this is also a certain category of representations for the Cherednik algebra of $\mathbb{Z}_n$).
- If $V = \operatorname{span}(1, \ldots, 1)$, then the hyperplane arrangement is n points on a line, and the associated category is a regular block of parabolic category $\mathcal{O}^{\mathfrak{p}}$ for $\mathfrak{sl}_n$, where $\mathfrak{p}$ is the parabolic preserving a line.

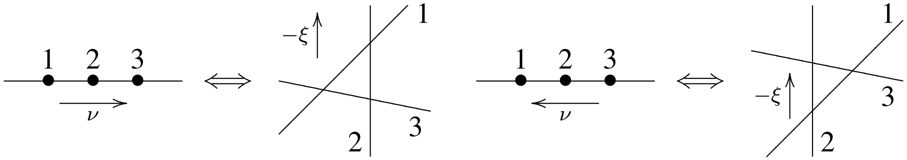
Note: these are Koszul dual!

Gale duality

There's a natural duality on the set of polarized hyperplane arrangements:

$$\mathcal{V} = (V \subset \mathbb{R}^n, \xi, \nu) \iff \mathcal{V}^\vee = (V^\perp \subset \mathbb{R}^n, -\nu, -\xi)$$

This correspondence is surprisingly hard to visualize, so here are some simple examples



Theorem (BLPW)

$(A(\mathcal{V}))^\star \cong A(\mathcal{V}^\vee)$

Derived equivalences

The fact that our result depends on the parameters ξ and ν is a bit dissatisfying. How can we compare the algebras for $\mathcal{V}$ and $\mathcal{V}' = (V, \xi', \nu')$?

Theorem (BLPW)

As along as all parameters are generic, we have an equivalence of derived categories $D(A(\mathcal{V})) \cong D(A(\mathcal{V}'))$, even though the algebras $A(\mathcal{V})$ and $A(\mathcal{V}')$ are generally not Morita equivalent.

These isomorphisms are not canonical at all. In fact, they seem to only be unique up to an action of $\pi_1(\mathsf{Pol}_{\mathbb{C}}(V))$, the complexification of the spaces of choices of polarization of V.

Probably this has something to do with stability conditions on this category or maybe a quotient of it.

Why?

So, these algebras have a really shocking amount of structure for some random relations we wrote down. What could possibly explain this?

If there are any experts in the audience on the Bernstein-Gelfand-Gelfand category $\mathcal{O}_{\mathfrak{g}}$, you might have noticed that the results above sound an awful lot like ones about $\mathcal{O}_{\mathfrak{g}}$.

One answer

Both categories can be realized as A-branes on a resolution of a symplectic cone!

- If you're an algebraist: an A-brane is a representation of a deformation quantization of functions on the cone.
- If you're a geometer: an A-brane is an object in the Fukaya category of said resolution.

Symplectic cones

When I say "symplectic," I mean *algebraically symplectic* with a $\mathbb{C}$-valued holomorphic 2-form ω.

From the perspective of a $\mathbb{R}$-symplectic geometer, $\Re(\omega)$ and $\Im(\omega)$ are two symplectic forms, related by the complex structure. In all cases we'll discuss, these are actually $2/3$ of a hyperkähler structure.

Algebraic symplectic implies Calabi-Yau, so it is very restrictive.

We'll be interested in a smooth symplectic variety $\tilde{X}$ which is a resolution of an affine cone X (i.e. X is an affine variety invariant under scaling). In this case, we say $\tilde{X}$ is a **symplectic resolution**.

Nilpotent cones

Let $\mathcal{N}_{\mathfrak{g}}$ be the cone of nilpotent elements in a complex Lie algebra $\mathfrak{g}$.

There's a symplectic resolution of singularities, the **Springer resolution**

$$\{(n,\mathfrak{b})|n \in \mathcal{N}, \mathfrak{b} \text{ a Borel}, n \in \mathfrak{b}\} = \tilde{\mathcal{N}} \cong T^*G/B \to \mathcal{N}\}.$$

The universal enveloping algebra of $\mathfrak{g}$ is a deformation quantization of $\mathcal{N}$, so the BGG category $\mathcal{O}$ obviously fits into the algebraic definition of A-branes I gave. For the geometric one, this is trickier, but a theorem:

Theorem (Beilinson-Bernstein, Nadler-Zaslow)

There is in an inclusion $(\mathcal{O}_{\mathfrak{g}})_0 \hookrightarrow \mathrm{Fuk}(T^*G/B)$.

Hypertoric varieties

What symplectic cone corresponds to a hyperplane arrangement?

Let T be a *complex* torus acting diagonally on $\mathbb{C}^n$. As always, we have a canonical moment map $\mu : T^*\mathbb{C}^n \to \mathfrak{t}^*$. Let $X/\!\!/_\alpha G$ denote the GIT quotient of a variety X for the character α.

One can do a symplectic reduction in the algebraic category

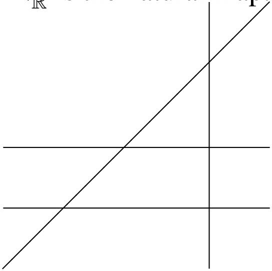
$$\mathfrak{M}_\alpha = \mu^{-1}(0)/\!\!/_\alpha T = \bigsqcup_{v \in \mathbb{C}^n} N^*(T \cdot v)/\!\!/_\alpha T$$

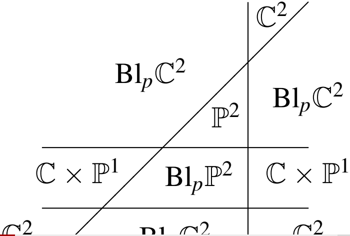
and obtain a **hypertoric variety**, closely tied to the combinatorics of T acting on V. $\mathfrak{M}_0$ is a cone, and $\mathfrak{M}_\alpha$ for α generic, is a symplectic resolution of $\mathfrak{M}_0$.

This can also be understood as a hyperkähler reduction for the compact form of T.

Hypertoric varieties and hyperplane arrangements

Our original data can be recovered as the affine hyperplane arrangement $(\ker \iota, \alpha, -)$ where $\iota : \mathbb{R}^n \to \mathfrak{t}^*_{\mathbb{R}}$ is the natural map.





$A(\mathcal{V})$ and geometry

The category $A(\mathcal{V}) - \mathsf{mod}$ for a polarized arrangement $\mathcal{V} = (V, \xi, \nu)$ has a geometric interpretation similar to that of $\mathcal{O}_{\mathfrak{g}}$.

Conjecture (BLPW)

$A(\mathcal{V}) - \mathsf{mod}$ *has a full and faithful inclusion into either interpretation of A-branes on* $\mathfrak{M}_\xi$*, with its image described by conditions similar to* $(\mathcal{O}_{\mathfrak{g}})_0$.

The algebraic construction is based on an algebra M_V, which we can construct by non-commutative Hamiltonian reduction of the algebra of differential operators $\mathcal{D}_{\mathbb{C}^n}$ by T.

This deformation quantization of $\mathfrak{M}_V$ can be regarded as an analogue of the universal enveloping algebra, and one can search for analogues of all results of Lie theory. But that's another talk.

Canonical deformation

If X is a cone with symplectic resolution of singularities $\tilde{X}$, then

Proposition (Kaledin-Verbitsky)

There is (roughly) a universal deformation $\tilde{Y}$ of $\tilde{X}$ as a symplectic variety over the base $H^2(\tilde{X})$.

- If $\tilde{X}$ is a hyperkähler quotient of T^*V by G, then this is simply given by the family of reductions at different complex moment map values.

$$\tilde{Y} = \bigsqcup_{v \in V} N^*([G,G] \cdot v) /\!\!/_{\alpha} G$$

- If $\tilde{X} = T^*G/B$, then $\tilde{Y} = G \times_B \mathfrak{b}$, the Grothendieck simultaneous resolution.
- If $\tilde{X} = \mathsf{Hilb}^n(\mathbb{C}^2)$, then the fibers of $\tilde{Y}$ are called the Calogero-Moser spaces.

Deformation quantization

Definition

For purposes of this talk, a deformation quantizations of a symplectic cone $\operatorname{Spec} R$ *is a filtered algebra* A *such that* $[A_r, A_s] \subset A_{r+s-m}$*, and* $\operatorname{gr} A \cong R$*, with the induced Poisson structure given by the reduction of commutator.*

There is a generalization of the canonical deformation that includes non-commutative deformations.

Proposition (Bezrukavnikov-Kaledin)

There is a canonical deformation quantization A_Y *of* Y *with center given by* $\operatorname{Sym}^*(H_2\tilde{X})$.

Alternatively, this can be seen as a family A_X^λ *of the deformation quantizations of* X *over* $H^2(\tilde{X})$.

Examples

Some pretty interesting algebras show up when we do this. A couple of them are quite familiar, but it also gives us some new and interesting algebras.

nilcone: $\mathcal{N}_{\mathfrak{g}}$	$\Longleftrightarrow$	universal enveloping algebra: $U(\mathfrak{g})$
symmetric power: $\mathrm{Sym}^n(\mathbb{C}^2)$	$\Longleftrightarrow$	rational Cherednik algebra U_c for S_n
affine Grassmannian slice: $\mathfrak{W}^{\lambda}_{\mu}$	$\overset{?}{\Longleftrightarrow}$	primitive quotient of shifted Yangian
quiver variety: $\mathfrak{Q}^{\lambda}_{\mu}$	$\Longleftrightarrow$	here be dragons
hypertoric variety: $\mathfrak{M}_{\mathcal{A}}$	$\Longleftrightarrow$	

"Dragons" is a slight exaggeration; we know what the algebras are, but as far as I know, there is no literature on them.

Category $\mathcal{O}$

For any symplectic cone, we have a class of categories of modules over the deformations quantization, which we can think of as analogues of $\mathcal{O}_{\mathfrak{g}}$.

- If you're an algebraist, you'll take modules locally finite for the action of the non-negative weight subalgebra for this $\mathbb{C}^*$-action.
- If you're a symplectic geometer, you'll take branes with particular "conditions at ∞" determined by the $\mathbb{C}^*$-action.

Conjecture (Too optimistic)

For each symplectic cone with a Hamiltonian $\mathbb{C}^*$-action (with suitable hypotheses), category $\mathcal{O}$ is Koszul, quasi-hereditary, and up to derived equivalence, depends only on the fixed points of the $\mathbb{C}^*$-action.

But what about the Koszul duality results? How can we generalize the relationship between Gale dual hypertoric varieties?

A strange duality

Based on various pieces of evidence, Braden, Licata, Proudfoot and I have suggested that this should reflect some kind of underlying duality between symplectic cones.

Conjecture

This is reflected by a Koszul duality between certain category $\mathcal{O}$'s attached to dual cones.

Observation

Our examples coincide with a notion of duality in physics; they are the Higgs branches of mirror dual 3-dimensional gauge theories.

OK, that's probably not very helpful (I'm not able to convert the physics into a mathematically rigorous definition), so let me give the examples.

Examples of duality

So here's the list of symplectic cones thus far that we believe we have found the dual to:

hypertoric variety: $\mathfrak{M}_{\mathcal{A}}$	$\Longleftrightarrow$	Gale dual: $\mathfrak{M}_{\mathcal{A}^\vee}$
nilcone: $\mathcal{N}_{\mathfrak{g}}$	$\Longleftrightarrow$	Langlands dual: $\mathcal{N}_{{}^L\mathfrak{g}}$
symmetric power: $\mathrm{Sym}^n(\mathbb{C}^2)$	$\Longleftrightarrow$	symmetric power: $\mathrm{Sym}^n(\mathbb{C}^2)$
G_I-instantons on $\widetilde{\mathbb{C}^2/\Gamma_J}$	$\Longleftrightarrow$	G_J-instantons on $\widetilde{\mathbb{C}^2/\Gamma_I}$
	$\Gamma_I \overset{\text{McKay}}{\longleftrightarrow} G_I$	
quiver variety: $\mathfrak{Q}_\mu^\lambda$	$\Longleftrightarrow$	affine Grass. slice: $\mathfrak{W}_\mu^\lambda$

Simplest interesting example: $T^*\mathbb{P}^{n-1} \Leftrightarrow \widetilde{\mathbb{C}^2/\mathbb{Z}_n}$ or, in terms of cones, $M_{n\times n}^{\mathrm{rk}\,1} \Leftrightarrow \mathbb{C}^2/\mathbb{Z}_n$.

A duality conjecture

Let G a reductive complex group with Cartan subgroup H. Let $C(H)$ be the cocharacter lattice of H, and let $\grave{\lambda}(t) = (\lambda(t), t^{-1}) \in C(H \times \mathbb{C}^*)$.

Conjecture

Each symplectic cone X with a Hamiltonian G-action commuting with the cone $\mathbb{C}^*$ action (thus giving a $G \times \mathbb{C}^*$-action) has a dual cone $X^\vee$ such that

- *There is a reductive group $G^\vee$ such that $(X^\vee)^\vee = X$.*
- *For each $\lambda \in C(H)/W$, we have a partial symplectic resolution $\tilde{X}^\vee_\lambda \to X^\vee$ equipped with an ample line bundle L_λ.*
- *$\tilde{X}^\vee_\lambda$ is an orbifold if and only if λ has isolated fixed points.*
- *For any $\lambda \in C(H)$ and $\mu \in C(H^\vee)$, we have a natural bijection between the components of fixed points of $\grave{\lambda}$ on $\tilde{X}_\mu$ and of $\grave{\mu}$ on $\tilde{X}^\vee_\lambda$.*

If you like hyperkähler structures, you can think that rather than a line bundle L_λ, we've picked a hyperkähler structure on $\tilde{X}^\vee_\lambda$ such that $[\omega_\mathbb{R}] = c_1(L_\lambda)$.

A duality conjecture

- For $X = \mathcal{N}_{\mathfrak{g}}$ and $X^\vee = \mathcal{N}_{^L\mathfrak{g}}$, a cocharacter $\lambda \in C(H)$ corresponds to the line bundle L_{λ^+} on $T^{*L}G/^LB$, where λ^+ is the dominant coweight in the orbit of λ. Thus

$$\tilde{X}^\vee_\lambda \cong \{(n,x) | x \in (^LG/^LP)^n\} \subset \mathcal{N}_{^L\mathfrak{g}} \times {}^LG/^LP$$

  where ^LP is the parabolic associated to λ.

- If $X = T^*\mathbb{C}^n /\!/\!/\!/_0 T$ is a hypertoric variety, then $G = H = (\mathbb{C}^*)^n/T$, and the Gale dual $X^\vee$ is a GIT reduction of $T^*\mathbb{C}^n$ by ^LH, which still carries an action of ^LT. Thus, if we pick $\lambda \in C(H)$, we have think of it as a weight of ^LH, and thus let

$$\tilde{X}^\vee_\lambda = T^*\mathbb{C}^n /\!/\!/\!/_{(0,0,\lambda)} {}^LH.$$

  The line bundle is the natural one we obtain from taking the GIT quotient. This is also the unique line bundle such that $[\omega_{\mathbb{R}}] = c_1(L_\lambda)$.

Equivariant cohomology of $\tilde{X}$

Assume $\lambda \in C(H)$ and $\mu \in C(H^\vee)$ have isolated fixed points, and that $\tilde{X}^\vee_\lambda, \tilde{X}_\mu$ are rationally smooth and equivariantly formal.

We call Y equivariantly formal for a torus H if pullback is an injection

$$H^*_H(Y) \hookrightarrow H^*_H(Y^H) \cong \bigoplus_{a \in Y^H} H_H(\{a\}).$$

Let R_Y be the subring of $H^*_H(Y)$ generated by H^2_H over H^0_H. More geometrically, we have

$$\operatorname{Spec} R_Y = \bigcup_{a \in Y^H} H^H_2(\{a\}) \subset H^H_2(Y).$$

That is, all information about R_Y is encoded in this subspace arrangement.

Examples

Variety $\tilde{X}$	Spec R	**Duality**
T^*G/B	$\bigcup_{w\in W}\Gamma_w \subset \mathfrak{t}^*\oplus\mathfrak{t}^*$	Langlands
$\mathfrak{M}_{\mathcal{V}}$	$\bigcup_{\beta \text{ a basis of } V^*}\mathbb{C}^\beta \subset \mathbb{C}^n$	Gale
$\mathsf{Hilb}^n(\mathbb{C}^2)$	$\bigcup_{\lambda\dashv n}(1,\mathrm{Con}(\lambda)) \subset \mathbb{C}^2$	self-dual

Observation (Goresky-MacPherson, BLPW)

There is an obvious "duality" on subspace arrangements, sending all subspaces to their annihilator. Let

$$R^\vee_{\tilde{X}} = \mathbb{C}\Big[\bigcup_{a\in\tilde{X}^H} H_2^H(\{a\})^\perp\Big] \subset H_2^H(\tilde{X})^*.$$

For all the examples above, we have a natural isomorphism $R^\vee_{\tilde{X}_\mu} \cong R_{\tilde{X}^\vee_\lambda}$.

GM duality for Koszul algebras

Interestingly, the same phenomenon holds for a general class of Koszul algebras, independent of any connection to geometry.

Any Koszul algebra A over an algebraically closed field k has a canonical flat deformation $\grave{A}$ over $Z(A^\star)_2$ the degree 2 part of the center of the dual $A^\star$.

Assume that A is quasi-hereditary, and the center $Z(\grave{A})$ is also flat.

Let R_A be the subalgebra of $Z(\grave{A})$ generated by $Z(\grave{A})_2$. As before, $\operatorname{Spec} R_A \subset Z(\grave{A}^\star)$ is a union of subspaces. Let $R_A^\vee$ be the coordinate ring of the union of the annihilators.

Theorem (BLPW)

$R_A^\vee = R_{A^\star}$

As a corollary, proving a categorical duality would imply the cohomological duality on the previous page.

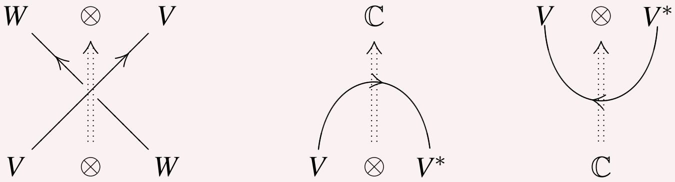
Knot homology

Let me now switch gears a bit, and talk about how this might help solve some mysteries in geometric knot homology. Now is your chance to flee if that doesn't suit you.

Reshetikhin-Turaev invariants

Reshetikhin and Turaev gave a mathematical construction of the invariants suggested by thinking about Chern-Simons theory (the definitions from a physics perspective involve path integrals).

We use the theory of quantum groups to attach maps to small diagrams like:



These are called the **braiding**, the **evaluation** and the **coevaluation**.

Composing these together for a given link results in a scalar: the **Reshetikhin-Turaev invariant** for that labeling.

Reshetikhin-Turaev invariants

So, what we'd like to find is

- categories $\mathcal{C}_{\lambda_1,\cdots,\lambda_n}$ such that

$$K_{\mathbb{C}}(\mathcal{C}_{\lambda_1,\cdots,\lambda_n}) \cong V_{\boldsymbol{\lambda}} = V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_n}$$

  where V_{λ_i} is the representation of G of highest weight λ.
- A **grading** on these categories (if $\mathcal{C} \cong R - \mathsf{mod}$, this is a grading on R). The Grothendieck group of the graded category can be thought of as $K_{\mathbb{C}[q,q^{-1}]}(\mathcal{C})$ where q is grading shift. This should be the representation $V_{\boldsymbol{\lambda}}$ of the quantum group $U_q(\mathfrak{g})$.
- Graded functors $\mathcal{E}_i, \mathcal{F}_i : \mathcal{C} \to \mathcal{C}$ that give the action of the quantum group. Actually, we will need derived categories (i.e. allow $\mathcal{E}_i(X)$ to be a complex in $\mathcal{C}$).
- Functors for the operations of braiding and (co)evaluation.

Reshetikhin-Turaev invariants

This has been done in the case of the representations $\wedge^i\mathbb{C}^n$ of $\mathfrak{sl}_n$ by Mazorchuk and Stroppel (building on work of a number of authors).

- They used certain categories of infinite dimensional representations of the Lie algebra $\mathfrak{sl}_m$.
- They recover knot invariants defined earlier by Khovanov and Rozansky. In particular, they give a new construction of Khovanov homology.

Work of Seidel-Smith and Manolescu realized the *same* invariants in terms of the Fukaya category of slices to nilpotent orbits.

Problem 1: We don't know how to associate categories to tensor products of arbitrary representations.

Problem 2: In a few particular cases, we have many ways of doing this, but don't know the relationship between them.

Problem 3: In some cases one uses modules over an algebra, sometimes one

Problem 3

While these constructions are in different languages, we expect that there is a single underlying construction which

- takes in a symplectic variety X with a Hamiltonian $\mathbb{C}^*$-action and
- outputs a unique category $\mathcal{Q}(X)$ (when X is a resolution of a cone).

1. If you're an algebraist, you'll see this as the category of representations of a deformation quantization of $\mathrm{Fun}(X)$ which are locally finite for a certain subalgebra.
2. If you're a symplectic geometer, you'll see this part of the Fukaya category of X where the Lagrangians have particular behavior "at ∞."
3. If you're a physicist, you'll see this as "A-branes" and the correspondence between 1 and 2 is that an enlargement of the Fukaya category contains a "canonical brane" whose endomorphisms are the deformation quantization.

However, Nadler and Zaslow have made this correspondence precise in the case of a cotangent bundle through a more "bare-handed" approach.

Problem 2

- Stroppel's construction of Khovanov homology is based on the the cotangent bundle to a Grassmannian $T^*\mathrm{Gr}_n^{2n}$ (parabolic category $\mathcal{O}_{\mathfrak{sl}_{2n}}^{\mathfrak{p}_i}$). The quantum group action and (co)evaluation are given by natural geometric functors.
- The Seidel-Smith construction uses the slice $S_{(n-i,i)}$ perpendicular to the nilpotent orbit of Jordan type (n, n) (category $\mathcal{O}$ for a finite W-algebra). The braiding action is given by the analogous functors.

How did this come to be?

Answer: (Braden, Licata, Proudfoot, W.)

These varieties are **symplectic dual**.

Symplectic duality

The pair $T^*\mathrm{Gr}_i^n$ and $S_{(n-i,i)}$ is among the varieties we suggested would be dual earlier.

As you've seen, we have a number of conjectures about connections between these varieties, but the one most relevant for us is

Conjecture (BLPW)

The categories $\mathcal{Q}(X)$ and $\mathcal{Q}(X^\vee)$ are Koszul dual.

This is a special equivalence of derived categories; in particular, we get a canonical isomorphism $K_{\mathbb{C}}(\mathcal{Q}(X)) = K_{\mathbb{C}}(\mathcal{Q}(X^\vee))$.

So, we have a very rich categorification story, since we can use geometry on both sides of duality.

Quiver varieties

For Problem 1, we know we want to construct a bunch of categories, $\mathcal{C}_{\boldsymbol{\lambda}}$, one for each sequence of weights $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_n)$ of the simple Lie group G.

In fact, we can take separate categories $\mathcal{C}_{\boldsymbol{\lambda}}^{\mu}$ for the different weight spaces.

There's a natural guess: For each pair of weights λ, μ, Nakajima has defined a quiver variety X_{λ}^{μ}. This is a symplectic variety, described as an algebraic symplectic quotient of a vector space.

There's a number of links between quiver varieties and representation theory; the simplest is that $H^{mid}(X_{\lambda}^{\mu}) \cong (V_{\lambda})_{\mu}$.

Quiver varieties

Even better, Nakajima has also defined a natural $\mathbb{C}^*$ action for each sequence $\boldsymbol{\lambda}$ such that $\lambda = \lambda_1 + \cdots + \lambda_n$.

Proposition (Nakajima)

There is an isomorphism $H^{mid}\left((X^{\mu}_{\boldsymbol{\lambda}})^{\mathbb{C}^*}\right) \cong (V_{\boldsymbol{\lambda}})_{\mu}$.

This cohomology group is a "first approximation" to the Grothendieck group of category $\mathcal{Q}$, so this is a good sign.

Theorem (Zheng)

$$K_{\mathbb{C}[q,q^{-1}]}(\mathcal{Q}(X^{\mu}_{\boldsymbol{\lambda}})) = (V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_n})_{\mu}$$

One can realize this as a category of modules over a noncommutative algebra A^{λ}_{μ} which is a deformation of $\mathrm{Fun}(X^{\mu}_{\lambda})$. The action of the quantum group and functors for (co)evaluation are given by tensoring with certain bimodules.

The affine Grassmannian

There is another class of varieties whose geometry is closely tied with the representations of simple Lie groups.

- Let $G((t))$ be the Laurent series points of G.
- Let $G[[t]]$ be the Taylor series points of G.
- Let $K = \{g \in G[t^{-1}] | g \equiv 1 \pmod{t^{-1}}\}$ be the subgroup complementary to $G[[t]]$.

The affine Grassmannian is the quotient $\mathsf{Gr} = G((t))/G[[t]]$.

The $G[[t]]$-orbits on Gr are indexed by dominant coweights of G. We let

$$G_\lambda = G[[t]] \cdot \lambda(t) \cdot G[[t]] \qquad \mathsf{Gr}_\lambda = G_\lambda / G[[t]].$$

For any sequence $\boldsymbol{\lambda}$ of weights, we have a variety

$$\mathsf{Gr}_{\boldsymbol{\lambda}} = \overline{G_{\lambda_1}} \times_{G[[t]]} \cdots \times_{G[[t]]} \overline{G_{\lambda_n}}/G[[t]] \qquad m_{\boldsymbol{\lambda}} : \mathsf{Gr}_{\boldsymbol{\lambda}} \to \overline{\mathsf{Gr}_\lambda}$$

Affine Grassmannians and shifted Yangians

The varieties $\mathsf{Gr}_{\boldsymbol{\lambda}}$ aren't symplectic, but they are a union of finitely many symplectic pieces (they're Poisson).

Given a sequence of coweights $\boldsymbol{\lambda}$ and another coweight λ, we get a new symplectic variety by looking at $\mathfrak{W}^{\mu}_{\boldsymbol{\lambda}} = m_{\boldsymbol{\lambda}}^{-1}(K \cdot \mu(t)) \subset \mathsf{Gr}_{\boldsymbol{\lambda}}$.

Unlike the varieties we've talked about earlier, this isn't smooth. This creates problems for us if we want to talk about its Fukaya category, but we can still hope it has a nice deformation quantization.

The varieties $\mathfrak{W}^{\mu}_{\lambda}$ are just the closures of symplectic leaves of $K \cdot \mu(t) \subset \mathsf{Gr}$, so really, we can quantize the whole thing, and then take quotients.

Conjecture

The shifted Yangian $Y_{\mu}(\mathfrak{g})$ is a deformation quantization of $K \cdot \mu(t)$. Category $\mathcal{Q}$ for a quotient $Y^{\boldsymbol{\lambda}}_{\mu}(\mathfrak{g})$ will categorify the μ-weight space of $V_{\boldsymbol{\lambda}}$. The braiding functors are given by tensor product with bimodules over the $Y_{\mu}(\mathfrak{g})$.

Knot invariants

If we could prove that we had Koszul dual categorifications arising from quiver varieties and the affine Grassmannian, then this would be enough for us to construct categorified R-T invariants.

But this Koszul duality statement looks rather hard, so our plan is to define the braiding functors without having to dualize.

This works in the $V_{\lambda_i} = \mathbb{C}^n$ case; one can use twisting functors.

Conjecture (work in progress with Stroppel)

There are analogues of twisting functors for the category $\mathcal{Q}(X^{\mu}_{\boldsymbol{\lambda}})$ attached to a quiver variety which give the braiding. These will be sent to the braiding functors under the Koszul duality with $\mathcal{Q}(\mathfrak{W}^{\mu}_{\boldsymbol{\lambda}})$.

These can be used to construct functorial knot invariants in combination with the (co)evaluation functors on $\mathcal{Q}(X^{\mu}_{\boldsymbol{\lambda}})$.