18.781 Problem Set 7 - Fall 2009

Due Thursday, Nov. 5 at 2:30

1. (Niven 7.1.1) Expand the following fractions into simple continued fractions:

   (a) $\frac{17}{3}$
   (b) $\frac{3}{17}$
   (c) $\frac{8}{1}$
   (d) $\frac{71}{34}$.

2. Prove that if $x = [a_0, a_1, \ldots, a_r]$ is greater than 1, then $\frac{1}{x} = [0, a_0, a_1, \ldots, a_r]$.

3. (Niven 7.1.3) Convert the continued fractions into rational numbers:

   (a) $[2, 1, 4]$
   (b) $[-3, 2, 12]$
   (c) $[0, 1, 1, 100]$.

4. (Niven 7.1.4 & 7.1.5) Suppose that $c > d$, and that all a_i are integers.

   (a) Prove that $[a_0, c] < [a_0, d]$.
   (b) Prove that $[a_0, a_1, c] > [a_0, a_1, d]$.
   (c) Prove that $[a_0, a_1, \ldots, a_r, c] < [a_0, a_1, \ldots, a_r, d]$ if and only if r is even, with the opposite (strict) inequality when r is odd.

5. Calculate the first three convergents for

   (a) e^2
   (b) 2π.

6. What are the elements of the Farey sequence of order 12 that are between 1/2 and 1/3?

7. (Niven 6.1.4) Let $a/b, c/d$ run over all pairs of consecutive fractions in the Farey sequence of order n. Show that

$$\min\left(\frac{c}{d} - \frac{a}{b}\right) = \frac{1}{n(n-1)} \qquad \text{and} \qquad \max\left(\frac{c}{d} - \frac{a}{b}\right) = \frac{1}{n}.$$

8. (Niven 6.1.9) For each fraction in reduced terms a/b, let $\mathcal{C}(a/b)$ denote the circle in the plane of radius $\frac{1}{2b^2}$ centered at $(\frac{a}{b}, \frac{1}{2b^2})$. These are called **Ford circles**.

   Show that the no point lies in the interior of two of these circles, and that if two circles $\mathcal{C}(a/b)$ and $\mathcal{C}(c/d)$ are tangent, then the corresponding fractions $a/b, c/d$ are consecutive in the Farey sequence of some order.

9. (Niven 7.3.2) Evaluate the infinite continued fractions $[2, 1, 1, \ldots,]$ and $[2, 3, 1, 1, 1, \ldots]$

10. (Niven 7.3.6) Let p be a prime with $p \equiv 1 \pmod 4$ and consider u such that $u^2 \equiv -1 \bmod p$. Consider the continued fraction expansion $u/p = [a_0, \ldots, a_n]$, and let $r_i = h_i/k_i$ be the last convergent such that $k_i < \sqrt{p}$. Using our approximation results, show that $x = k_i$ and $y = h_i p - u k_i$ are integers which satisfy $x^2 + y^2 = p$.