18.781 Problem Set 3 - Fall 2009

Due Thursday, Oct. 1 at 2:30

1. Simultaneously solve the congruences

$$x \equiv 2 \pmod 4, \qquad x \equiv 5 \pmod 3, \qquad x \equiv 3 \pmod 7.$$

2. (Niven 2.3.8) Find the smallest positive integer whose remainder is $1, 2, 3, 4$, and 5 when divided by $3, 5, 7, 9$, and 11, respectively. What is the second smallest such integer?

3. (Niven 2.3.18) For any $k \geq 1$, prove that there exist k consecutive positive integers that are each divisible by a square number. For example, the sequence $\{48, 49, 50\}$ works for $k = 3$.

4. (Niven 2.1.51) Prove that

$$(p-1)! \equiv p-1 \pmod P,$$

where $P = 1 + 2 + \cdots + p - 1$.
Hint: Use the Chinese Remainder Theorem and Wilson's Theorem.

5. (Niven 2.3.20) Prove that there is a simultaneous solution of $x \equiv a_1 \pmod{m_1}, x \equiv a_2 \pmod{m_2}$ iff $a_1 \equiv a_2 \pmod{(m_1, m_2)}$. Prove that the solution is unique modulo $[m_1, m_2]$.

6. (Niven 2.3.25) Prove that the number of integers $1 \leq n \leq mk$ that satisfy $(n, m) = 1$ is $k\phi(m)$.

7. (Niven 2.3.34) Prove that there is no solution of the equation $\phi(x) = 14$ and that 14 is the least positive even integer with this property. Apart from 14, what is the next smallest even integer such that $\phi(x) = n$ has no solution?

8. (Niven 4.2.4) Find the smallest m for which there exists another $n \neq m$ with $\sigma(m) = \sigma(n)$.

9. (Niven 4.2.5) Prove that

$$\prod_{d|n} d = n^{d(n)/2}.$$

10. (Niven 4.2.9) Suppose that $f(n)$ and $g(n)$ are multiplicative.
    (a) Prove that $F(n) := f(n)g(n)$ is also multiplicative.
    (b) If $g(n) \neq 0$ for all n, prove that $G(n) := f(n)/g(n)$ is multiplicative.

11. (Niven 4.2.12) Prove that $d(n) = \#\{d \mid n\}$ is odd iff n is a square.

12. (Niven 4.2.16 and 4.2.19) A positive integer n is a perfect number if $\sigma(n) = 2n$ (i.e., n is the sum of its proper divisors; for example, $6 = 1 + 2 + 3$ is perfect). Prove that if $2^m - 1 = p$ is prime, then $2^{m-1}p$ is perfect.

(Bonus) Prove that every even perfect number has this form.

Remark: It is widely believed that there are no odd perfect numbers, although this is still an open conjecture!

13. (Niven 4.2.1) Find n such that $\mu(n) + \mu(n+1) + \mu(n+2) = 3$.

14. (Niven 4.2.2) Prove that $\mu(n)\mu(n+1)\mu(n+2)\mu(n+3) = 0$ for all n.

15. Define
$$Sq(n) := \begin{cases} 1 & \text{if } n \text{ is square,} \\ 0 & \text{otherwise.} \end{cases}$$
Use Möbius inversion to find $f(n)$ such that
$$Sq(n) = \sum_{d|n} f(d).$$
Prove that f is multiplicative and find an explicit formula for $f(n)$.

(Bonus) The *Riemann Zeta function* is one of the most important functions in number theory (and the subject of a million dollar research prize!). It is defined for complex arguments s as
$$\zeta(s) := \sum_{n\geq 1} \frac{1}{n^s} = \prod_{p \text{ prime}} \frac{1}{1 - \frac{1}{p^s}},$$
although the above formulas only converge for $\Re(s) > 1$.

(a) Prove that the sum and product formulas for $\zeta(s)$ are actually equal.

(b) Can you find a similar formula for
$$g(s) = \sum_{n\geq 1} \frac{f(n)}{n^s}$$
for any totally multiplicative function f? What if f is just multiplicative?