18.781 Problem Set 1 - Fall 2009

Due Thursday, Sep. 17 at 2:30

Notation: PT=Pythagorean Triple, PPT=Primitive Pythagorean Triple

1. (Niven 5.3.3) Find all PT's whose terms form an
   (a) Arithmetic progression
   (b) Geometric progression
2. (Niven 5.3.7) For which n are there solutions to $x^2 - y^2 = n$?
3. (Niven 5.3.9) Prove that any integer n can be expressed in the form
$$n = x^2 + y^2 - z^2$$
   (contrast this with Gauss' 3-squares Theorem!).
4. Find all PPT's with $c = a + 2$.
5. The n-th *triangular number* is given by $T_n := 1 + 2 + \cdots + n$. The first few values are $T_1 = 1, T_2 = 3, T_3 = 6, \ldots$.
   (a) Prove using induction that $T_n = \frac{n(n+1)}{2}$.
   (b) Prove that for any n there is a PPT containing $4T_n$. For example, $(7, 24, 25)$ contains $4T_3 = 24$.
6. Prove that
$$\sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{n}{n+1}.$$
7. (Niven 1.2.2) Find the greatest common divisor $g = (1819, 3587)$, and find x, y such that
$$1819x + 3587y = g.$$
8. (Niven 1.2.9) Show that if $ac \mid bc$, then $a \mid b$.
9. (Niven 1.2.10) Show that if $a \mid b$ and $c \mid d$, then $ac \mid bd$.
10. (Niven 1.2.11) Prove that $4 \nmid (n^2 + 2)$ for any n.
11. (Niven 1.2.12) Given that $(a, 4) = (b, 4) = 2$, prove that $(a + b, 4) = 4$.
12. (Niven 1.2.17) Evaluate $(n, n+1)$ and $[n, n+1]$ ($[n, n+1]$ means "least common multiple of n and $n + 1$").
13. (Niven 1.2.36) Prove that $(a, b, c) = ((a, b), c)$.
14. (Niven 1.2.43) Prove that $a \mid bc$ if and only if $\frac{a}{(a,b)} \mid c$.
15. Prove that in the Euclidean algorithm, $r_{i+2} < \frac{1}{2} r_i$.

(Bonus) Find a bound on the total number of steps in the algorithm.

16. (Niven 1.2.45) Prove that any positive integer a can be uniquely expressed as
$$a = 3^m + b_{m-1}3^{m-1} + \cdots + b_1 3 + b_0,$$
   where $b_i = 0, 1,$ or -1.