

A strange duality and its implications for knot homology

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- 1 Strange duality
 - Not a definition
 - Examples
 - Common features
- 2 Cohomological duality
 - Goresky-MacPherson duality
 - MV cycles and core components
- 3 Categorical duality
 - Category \mathcal{O}
 - Koszul duality
- 4 Knot invariants

A strange duality

Braden, Licata, Proudfoot and I have observed that certain hyperkähler varieties seem to come in “dual pairs,” often reflecting some already known duality on combinatorial/representation theoretic objects connected to the varieties.

We have no precise definition of what it means for two varieties to be dual in the sense. There is a prescription from physics, which neither the physicists nor I are able to cast in mathematical language.

In lieu of a definition, let me provide some examples.

Flag varieties

If we let G be a compact Lie group, then the cotangent bundle $T^*\mathfrak{B}$ to the flag variety $\mathfrak{B} = G/T$ carries a holomorphic symplectic structure.

The moment map $\mu_{\mathbb{C}} : T^*\mathfrak{B} \rightarrow \mathfrak{g}_{\mathbb{C}}^*$ is in fact a resolution of singularities for its image, the cone \mathcal{N} of nilpotent elements in $\mathfrak{g}_{\mathbb{C}}^* \cong \mathfrak{g}_{\mathbb{C}}$, called **the Springer resolution**.

There is a Langlands dual group to G : this is the unique compact group whose torus is the dual torus ${}^L T = \text{Hom}(\pi_1(T), U(1))$, with the induced action of the Weyl group W . On the level of Lie algebras, this just reverses all arrows in the Dynkin diagram.

The dual $(T^*\mathfrak{B})^{\vee}$ is the corresponding space for the Langlands dual group $T^{*L}\mathfrak{B} = T^{*L}G/{}^L T$.

Slodowy slices

If $\mathcal{O}_\lambda, \mathcal{O}_\mu \subset \mathcal{N}_{\text{Mat}(n,n)}$ are two nilpotent orbits, where λ, μ are the partitions given by the Jordan blocks, such that $\mathcal{O}_\mu \subset \overline{\mathcal{O}_\lambda}$, the **Slodowy slice** is the variety $\mathcal{S}_\mu^\lambda = E \cap \overline{\mathcal{O}_\lambda}$ where E is an affine subspace transverse to \mathcal{O}_μ s.t. $\mathcal{O}_\mu \cap E = \{e\}$.

There is a resolution $T^*G/P_{\lambda^t} \rightarrow \overline{\mathcal{O}_\lambda}$ (not true in other types!). Restricting to \mathcal{S}_μ^λ gives a symplectic resolution $\widetilde{\mathcal{S}}_\mu^\lambda \rightarrow \mathcal{S}_\mu^\lambda$, which we also call the Springer resolution.

The dual $(\widetilde{\mathcal{S}}_\mu^\lambda)^\vee = \widetilde{\mathcal{S}}_{\lambda^t}^{\mu^t}$ is given by the Slodowy slice for the transposed partitions.

Hilbert schemes

For any smooth complex surface S and integer n , we have the **Hilbert scheme**

$$\mathrm{Hilb}^n(S) = \{\mathcal{I} \subset \mathcal{O}_S \mid \dim \mathcal{O}_S/\mathcal{I} = n\}.$$

This has a natural structure as a smooth projective variety, and is equipped with a natural resolution of singularities given any resolution $S \rightarrow S'$:

$$\mathrm{HC} : \mathrm{Hilb}^n(S) \rightarrow \mathrm{Sym}^n(S') = S'^n/\mathfrak{S}_n.$$

We will be particularly interested in the case of the affine surfaces $S' = \mathbb{C}^2/\Gamma$ for $\Gamma \subset SL_2(\mathbb{C})$, and $S = \widetilde{\mathbb{C}^2/\Gamma}$ the unique crepant resolution.

The dual $(\mathrm{Hilb}^n(\widetilde{\mathbb{C}^2/\Gamma}))^\vee$ is the space of G -instantons of charge n on \mathbb{C}^2 , where Γ and G are matched under the McKay Correspondence.

Hypertoric varieties

If one has a torus T acting on \mathbb{C}^n diagonally, there is a T -invariant hyperkähler structure on T^*V . That is, T^*V carries complex structures I, J, K , compatible symplectic structures $\omega_I, \omega_J, \omega_K$, with moment maps $\mu_I, \mu_J, \mu_K : T^*V \rightarrow \mathfrak{t}$.

One can do a hyperkähler reduction, using all three moment maps

$$\mathfrak{M}_\alpha = \mu_I(\alpha_I) \cap \mu_J(\alpha_J) \cap \mu_K(\alpha_K) / T.$$

and obtain a **hypertoric variety**, closely tied to the combinatorics of T acting on V .

The dual $\mathfrak{M}_{(0,0,\alpha)}^\vee$ is another hypertoric variety, that associated to the dual action of $L(D/T)$ on \mathbb{C}^n . This operation on the underlying combinatorial data is called **Gale duality**.

Quiver varieties

Given a quiver I , and λ, μ in the weight lattice of the corresponding Kac-Moody algebra \mathfrak{g}_I , Nakajima shows how to construct a **quiver variety** \mathfrak{Q}_μ^λ (also as a hyperkähler quotient).

This variety has very interesting connections to representation theory. In particular, $H^{mid}(\mathfrak{Q}_\mu^\lambda) \cong (V_\lambda)_\mu$, the μ -weight space of the irreducible representation V_λ of \mathfrak{g}_I of highest weight λ .

Assume that I is Dynkin (ADE). Then we have an affine Grassmannian $\text{Gr} = G((t))/G[[t]]$ for the group G associated to I , with $G[[t]]$ -orbits Gr_λ indexed by dominant weights of \mathfrak{g}_I . We have $\text{Gr}_\mu \subset \overline{\text{Gr}_\lambda}$ if $(V_\lambda)_\mu \neq 0$.

The dual $(\mathfrak{Q}_\mu^\lambda)^\vee$ is the slice \mathcal{W}_μ^λ to a point in Gr_μ in $\overline{\text{Gr}_\lambda}$.

Common features

Let me collect the properties of these varieties that will prove important in our discussion, with examples.

- 1 There is a resolution of singularities $\pi : X \rightarrow Y$, with Y an affine cone with vertex y_0 .
 - for $T^*V \mathbb{I} \mathbb{I} \mathbb{I} (0,0,\alpha)G$, given by natural map to $T^*V \mathbb{I} \mathbb{I} \mathbb{I} (0,0,0)G$.
 - for $T^*\mathfrak{B}$, \mathcal{S}_μ^λ , given by the Springer resolution; for $\text{Hilb}^n(\widetilde{\mathbb{C}^2/\Gamma})$, by the Hilbert-Chow map HC.
- 2 The variety X is exact holomorphic symplectic for $\omega_I + i\omega_J$, with a third non-exact symplectic form ω_K giving a hyperkähler structure.
- 3 X is one fiber of the family of hyperkähler varieties over $U \subset H^2(X, \text{Im } \mathbb{H})$.
 - for $T^*V \mathbb{I} \mathbb{I} \mathbb{I} G$, given by varying moment maps μ_I, μ_J, μ_K .
 - for $T^*\mathfrak{B}$, given by $G \times_B \mathfrak{b} \rightarrow \mathfrak{t}$. Generic fiber: coadjoint orbit.
 - for $\text{Hilb}^n(\widetilde{\mathbb{C}^2/\Gamma})$, this is “Calogero-Moser space.”

More common features

One of the most important features is a well-behaved action of a torus S .

- 4 X has a $\mathbb{C}^* \times S$ action which acts on $\omega_{\mathbb{C}}, \omega_{\mathbb{R}}$ by

$$a_{(t,s)}^* \omega_{\mathbb{C}} = t^m \omega_{\mathbb{C}} \quad a_{(t,s)}^* \omega_{\mathbb{R}} = \omega_{\mathbb{R}}.$$

- for $T^*V // G$, given by \mathbb{C}^* acting as dilation of cotangent fibers times a maximal torus of $\text{Aut}_G(V)$.
 - For $T^*\mathfrak{B}$, given by cotangent dilation times the maximal torus $T \subset G$.
 - for $\widetilde{\text{Hilb}^n(\mathbb{C}^2/\Gamma)}$, given by scalar multiplication on \mathbb{C}^2 times the torus $D \cap C_{SL_2(\mathbb{C})}(\Gamma)$.
- 5 This action contracts X to the projective Lagrangian subvariety $L = \pi^{-1}(y_0) \subset X$.
- for $T^*\mathfrak{B}$, $L = \mathfrak{B}$
 - for $\widetilde{\mathfrak{S}}_{\mu}^{\lambda}$, L is the Spaltenstein variety (Springer fiber if $\lambda = (n)$).
 - for a hypertoric variety \mathfrak{M} , we get a union of toric varieties (and \mathfrak{M} is a plumbing of their cotangent bundles).

A duality conjecture

Fix such an affine symplectic cone singularity Y with Hamiltonian torus S acting.

Conjecture

There is another such singularity Y^\vee with Hamiltonian torus S^\vee , such that there is a bijection between

- *pairs consisting of a partial resolution $X \rightarrow Y$, and a relatively ample line bundle \mathcal{L} on X .*
- *cocharacters $\mathbb{C}^* \rightarrow S^\vee$*

and vice versa. A cocharacter acting on a smooth resolution has isolated fixed points if and only if its dual partner does, and in this case there is a bijection between fixed points.

For any resolution, there is an isomorphism $\text{Pic}(X) \cong \text{Hom}(\mathbb{C}^, S^\vee)$.*

This has been checked for the examples mentioned earlier.

An observation from physics

All the singular hyperkähler cones in our picture also appear in physics.

They are Higgs branches of $N = 4$ supersymmetric 3-d field theories.

Remarkably, the dual pairs we listed above are attached to field theories which are “mirror dual” in a certain sense (very different from the mirror symmetry discussed by mathematicians, a 2-d phenomenon).

Some of the features we mentioned are natural from this perspective. The isomorphism $\text{Pic}(X) \cong \text{Hom}(\mathbb{C}^*, S^\vee)$ is the switching of mass and Fayet-Iliopolous parameters, and bijection between fixed points is that for supersymmetric vacua.

Others, like the duality conjectures we state later, seem not to yet have physical interpretations.

Equivariant cohomology of X

Assume for now that S has isolated fixed points on X , and X is equivariantly formal. That is, we have an injection

$$R_X = H_S^*(X) \rightarrow H_S^*(X^S) \cong \bigoplus_{a \in X^S} H_S(\{a\}).$$

More geometrically, we have

$$\text{Spec } R_X = \bigcup_{a \in X^S} H_2^S(\{a\}) \subset H_2^S(X).$$

That is, all information about $H_S^*(X)$ is encoded in this subspace arrangement.

There is an obvious “duality” on subspace arrangements, sending all subspaces to their annihilator. Let

$$R_X^\vee = \mathbb{C} \left[\bigcup_{a \in X^S} H_2^S(\{a\})^\perp \right] \subset H_2^S(X)^*.$$

Goresky-MacPherson duality

Proposition (Goresky-MacPherson, BLPW)

In all examples above where S has isolated fixed points, S^\vee acting on X^\vee does as well and

$$R_X^\vee = R_{X^\vee}.$$

Recent work of BLPW shows that a similar statement holds for the centers of natural deformations of certain dual Koszul algebras. This implies that GM duality would be a consequence of a deeper statement about Koszul duality between subcategories of the Fukaya category of X and X^\vee . More on that later.

Core components

Since X contracts onto L , we have $H_{mid}(X) \cong H_{top}(L)$, which has a canonical basis given by the classes of the components of L .

- For \mathfrak{Q}_μ^λ , this is the semi-canonical basis of $(V_\lambda)_\mu$.
- For $\widetilde{\mathfrak{S}}_\mu^{(n)}$, $H_{top}(L)$ is canonically isomorphic to the representation W_μ of \mathfrak{S}_n , and the components give a canonical basis on this representation.
- For \mathfrak{M} , $\dim H_{top}(L)$ is the top h -number of the corresponding matroid.

This vector space with basis is an important invariant of our symplectic singularity.

MV cycles

Now, consider the singular cone Y . Pick a generic cocharacter $\xi : \mathbb{C}^* \rightarrow S$, and let

$$Y_\xi = \{y \in Y \mid \lim_{t \rightarrow \infty} \xi(t) \cdot y = y_0\}.$$

Definition

*The components of Y_ξ are called **Mirković-Vilonen cycles**.*

Like the core components, the MV cycles correspond to a basis of $H_{top}^{BM}(Y_\xi)$, though they can also be thought of as giving a basis of a different cohomological vector space, the intersection cohomology $\mathbb{H}^*(Y)$.

There is an alternative extension of the theory of the cohomology of smooth varieties to singular ones, called intersection cohomology $\mathbb{H}^*(-)$. This preserves some good features of cohomology (Poincaré duality) but loses some (homotopy invariance).

Cohomological duality I

Given $\xi \in \text{Hom}(\mathbb{C}^*, S^\vee)$, we obtain (by duality) a partial resolution $X \rightarrow Y$.

Proposition

In each of our examples, there is a natural isomorphism between $\mathbb{H}(Y^\vee)$ and $H_{\text{mid}}(X)$ which induces a bijection between the components of L and the MV cycles of Y_ξ^\vee .

Perhaps the most interesting case is that of the quiver variety \mathcal{Q}_μ^λ and slice in the affine Grassmannian \mathcal{W}_μ^λ , where

$$H_{\text{mid}}(X) \cong (V_\lambda)_\mu \cong \mathbb{H}(\mathcal{W}_\mu^\lambda).$$

Note that while these spaces have geometric bases, we don't always expect the isomorphism to preserve them. For example, the core components of \mathcal{Q}_μ^λ give the semi-canonical basis, while MV cycles in \mathcal{W}_μ^λ give the canonical basis.

Cohomological duality II

This is the most degenerate case of a more general statement. Given a pair of cocharacters ξ, ξ^\vee , by duality, we obtain partial resolutions X, X^\vee . Let $X_\xi = \pi^{-1}(Y_\xi)$.

Conjecture

There is an isomorphism $H_{top}^{BM}(X_\xi) \cong H_{top}^{BM}(X_{\xi^\vee}^\vee)$, which induces a bijection between the components of these varieties.

As we mentioned, the component bases aren't quite the right ones here. We believe there is another basis given by the supports of representations of a deformation quantization of X which behaves better under duality.

Symplectically, this has something to do with what happens when we try to deform core components to smooth Lagrangian submanifolds of X .

Health warning

Take everything I say about the Fukaya category with a pinch of salt. I seriously doubt that I am in the top half of the people in this room in terms of understanding of it.

Basically, everything I say about the Fukaya category will actually be translations of statements about a category of representations of a deformation quantization of X , which I believe to be equivalent (by an analogue of Nadler-Zaslow).

Of course, this equivalence is very conjectural, so one must account for both the possibility of translation errors, and the possibility of an entire lack of a dictionary.

Category \mathcal{O}

While the Fukaya category of (X, μ_I) is a quite interesting invariant, we will only be interested in a small piece of it. I would be very interested if any of you symplectic types have a good explanation for it.

Definition

We say a Lagrangian K has property \mathcal{O} for ξ if

$$\lim_{t \rightarrow 0} (t, 1) \cdot K \subset X_\xi = \pi^{-1}(Y_\xi).$$

We let the category \mathcal{O}_ξ^X be the subcategory of the Fukaya category given by objects supported on Lagrangians with property \mathcal{O} .

We have a natural map $K^0(\mathcal{O}_\xi^X) \rightarrow H_{top}^{BM}(X_\xi)$, given by the multiplicity with which $\lim_{t \rightarrow 0} [(t, 1) \cdot K]$ wraps around a given component. Thus, \mathcal{O}_ξ^X is (hopefully) a categorification of $H_{top}^{BM}(X_\xi)$.

Koszul duality

For an abelian category A , we let Σ_A denote the set of simples, and Π_A denote the set of indecomposable projectives.

Definition

The categories A and B are Koszul dual if

$$\mathrm{End}_B(\oplus_{P \in \Pi_B} P) = \mathrm{Ext}_A^\bullet(\oplus_{S \in \Sigma_A} S),$$

- Koszul duality is symmetric.
- It's almost true that $D^b(A) \cong D^b(B)$ (true for graded enhancements).

Categorical duality

Conjecture

The categories \mathcal{O}_ξ^X and $\mathcal{O}_{\xi^\vee}^{X^\vee}$ each carry a t -structure such that the hearts are Koszul dual.

Theorem (BLPW)

This holds for hypertoric varieties (if ξ and ξ^\vee have isolated fixed points).

Theorem (Soergel, Nadler-Zaslow)

This holds for $T^\mathfrak{B}$.*

Twisting and shuffling

On the restriction to $U' = U \cap \langle J, K \rangle \subset U \subset \text{Im } \mathbb{H}$, we have $[\omega_I] = 0$. By the Moser trick, for any path between q and q' , we get a symplectomorphism $(X_q, \omega_I) \cong (X_{q'}, \omega_I)$, unique up to isotopy.

This induces a map $\pi_1(U') \rightarrow \pi_0(\text{Symp}(X, \omega_I)) \rightarrow \text{Aut}(\text{Fuk}(X, \omega_I))$.

Since I'm a representation theorist, I call these **shuffling functors**.

Conjecture

*Under the duality conjectured above, these functors are sent to **twisting functors** defined in terms of iterated projections from $\text{Fuk}(X, \omega_I)$ to $\mathcal{O}_{\xi'}^X$ followed by inclusions for different choices of ξ' .*

Geometric knot homology

There are two flavors of geometric knot homology which mysteriously seem to produce the same invariants:

- The work of Seidel-Smith on Khovanov homology and Manolescu on Khovanov-Rozansky homology related knot homology to the action of shuffling (monodromy) functors on Fukaya categories of Slodowy slices.
- The work of Stroppel on Khovanov-Rozansky homology related the *same* knot homologies to the action of rather odder functors (from the symplectic perspective) on the Fukaya categories of the cotangent bundles to partial flag manifolds.

These are exactly the sort of Fukaya categories related by our strange duality!

Knot homology (not) via the affine Grassmannian

The main issue is this: in order to categorify Reshetikhin-Turaev invariants of knots, one should categorify the tensor product of quantum group representations.

So, one approach to this would be to read the papers of Seidel-Smith and Manolescu, remember that Slodowy slices are special cases of slices to orbits in the affine Grassmannian, and to attempt to do the necessary categorification by playing with the appropriate Gr_λ 's.

This was done by Cautis and Kamnitzer, but the fact that almost all Gr_λ 's are non-smooth stymied them; they use different language, but at the end of the day, you want to use the Fukaya category of something that isn't smooth.

Knot homology via quiver varieties

Except, one doesn't really need the whole Fukaya category, just the category \mathcal{O} part of it, and *that* we have another way of thinking about: it's the same as category \mathcal{O} for a quiver variety, which *is* smooth.

Conjecture

The twisting functors on category \mathcal{O} for a quiver variety categorify the braiding on the category of quantum group representations, and other geometric functors categorify the evaluation and coevaluation.

This gives a categorified version of the Reshetikhin-Turaev invariant, and in particular, a bi-graded vector space for every knot labeled with representations of \mathfrak{g} for any simple Lie group.

The categories are already essentially constructed (using a somewhat different perspective by Zheng), but a lot of details remain to sort out on constructing the functors.

Thanks, y'all.