A geometric description of colored HOMFLYPT homology

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References:

This slide show can be downloaded from

`http://math.mit.edu/~bwebster/talks.html`

Some references:

- BW and GW, *Colored HOMFLYPT homology via equivariant sheaves and the weight filtration.* (`http://math.mit.edu/~bwebster/colorHOMFLYPT.pdf`)
- BW and GW, *A geometric model for the Hochschild homology of Soergel bimodules.*
- M. Mackaay, M. Stosic, P. Vaz, *The 1,2-coloured HOMFLY-PT link homology*
- M. Khovanov, *Triply-graded link homology and Hochschild homology of Soergel bimodules.*
- J. Bernstein and V. Lunts, *Equivariant sheaves and functors.*

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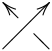
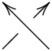
First, we associate a **shuffling (or Rouquier) complex** $F(\sigma)$ of $R - R$-bimodules to each braid σ, by assigning to elementary twists



$$\overset{F}{\mapsto} \quad R \otimes_{R^{s_i}} R \xrightarrow{\pi} R$$



$$\overset{F}{\mapsto} \quad R(2) \xrightarrow{\nu} R \otimes_{R^{s_i}} R$$

and let the composition of braids be given by tensor product of complexes: $F(\sigma\sigma') = F(\sigma) \otimes_R F(\sigma)$.

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$$H^*_{T\times T}(B) \cong R \qquad H^*_{T\times T}(P_i) \cong R \otimes_{R^{s_i}} R$$

and the maps in the complexes $F(\sigma_i^{\pm})$ are the pushforward and pullback maps in cohomology.

Boundary maps

If X is a n-manifold and a Y a subspace, we can define complexes of sheaves C_Y^* on X by

$$C_Y^*(U) = \cdots \to C^i(U \cap Y) \to C^{i+1}(U \cap Y) \to \cdots$$

where $C^*(U)$ denotes the space of singular cochains in U.

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Proposition

The map induced by the boundary map $\partial_* : \mathbb{H}_{T\times T}^*(B)[-2] \to \mathbb{H}_{T\times T}^*(P_i)$ *is equal to* ν*, the map in the shuffling complex.*

An analogous description exists for π, using the Verdier dual $\mathbb{D}C_Y^* \cong C_Y^!$.

HOMFLY and Hecke

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The functors $C^*_{P_i \backslash B} \star -$ give a braid group action on this category, categorifying the braid group action on the Hecke algebra. So, for a general braid $\sigma = \sigma_{i_1}^{\epsilon_1} \cdots \sigma_{i_m}^{\epsilon_m}$, we should consider the sheaf

$$C^*_\sigma = \left(C^*_{P_{i_1} \backslash B}\right)^{\epsilon_1} \star \cdots \star \left(C^*_{P_{i_m} \backslash B}\right)^{\epsilon_m}$$

and do something systematic that generalizes what we did to $C^*_{P_i \backslash B}$.

Weights and purity

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Theorem

There's a unique **weight filtration** *on C^*_Y for any variety Y which on the stalks is compatible with the weight grading on the stalk (any stalk of C^*_Y at $x \in X$ is of the form $H^*(U_x \cap Y)$).*

The chromatographic complex

For any complex of sheaves $\mathcal{F}$ with a filtration $\mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots \subset \mathcal{F}$ has an associated **chromatographic complex** $\mathsf{Chr}(\mathcal{F})$ given by

$$\cdots \to \mathcal{F}_i/\mathcal{F}_{i-1}[-i] \xrightarrow{\partial} \mathcal{F}_{i+1}/\mathcal{F}_i[-i-1] \to \cdots$$

where ∂ is the boundary map for $\mathcal{F}_i/\mathcal{F}_{i-1} \to \mathcal{F}_{i+1}/\mathcal{F}_{i-1} \to \mathcal{F}_{i+1}/\mathcal{F}_i$.

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Theorem

The shuffling complex $F(\sigma)$ is the equivariant cohomology $\mathbb{H}^*_{T\times T}(\mathsf{Chr}(C^*_\sigma))$ of the chromatographic complex of the sheaf C^*_σ.

For one crossing, this is just the fact that the weight filtration on $C^*_{P_I\backslash B}$ is the one we used before (which follows from our computation for $\mathbb{C}^*$).

Hochschild homology

Hochschild homology also naturally appears in the construction of KR homology. It is the derived functor of $HH^0(M) = M/[R, M]$ where as usual

$$[R, M] = R \cdot \{r \cdot m - m \cdot r | r \in R, m \in M\} \cdot R \subset M.$$

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For any projective resolution of M,

$$\mathbf{P}^\bullet = \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow 0$$

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Note that $HH^*(M)$ has the obvious "Hochschild" grading (which is independent of any grading on R) and another "polynomial" grading which arises from using a graded projective resolution of M.

Hochschild homology as extension of scalars

Consider an $R-R$ bimodule M as an $R \otimes R$ module. Note that the R-module $M/[R,M]$ can be rewritten as the extension of scalars

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Hochschild homology can thus be interpreted geometrically for sheaves that are pure (trivial weight filtration)

$$HH^*(\mathbb{H}^*_{T\times T}(SL_n;\mathcal{F}))\cong\mathbb{H}^*_{T_\Delta}(SL_n;\mathcal{F})$$

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HOMFLYPT homology is given by the complex $\mathbb{H}^*_{T_\Delta}(\mathsf{Chr}(C^*_\sigma))$.

The two gradings on Hochschild homology are given the weight and usual gradings on cohomology.

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Incidentally, this perspective simplifies the invariance proofs, so it's not entirely fanciful.

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$$\begin{tikzcd}
 & \{\text{braids}\} \arrow[dl, "F(\sigma)"'] \\
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\end{tikzcd}$$

And you might ask, “Where did that all get us? Bimodules were confusing enough!”

Generalization to colored HOMFLYPT

Let $\boldsymbol{\sigma}$ be a braid whose strands are colored by integers. We wish to construct the colored HOMFLY homology of $\boldsymbol{\sigma}$, which packages together the quantum invariants for knots colored by $\bigwedge^i \mathbb{C}^n$ for all $\mathfrak{sl}_n$.

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From a geometric perspective, this generalization is quite easy. Just replace B (resp. T) everywhere above with block UT (resp. block diagonal) matrices where the sizes of the blocks are the colorings. There is a geometric "cabling/projection" formula that let's us reduce everything to the original case.

b

Translation to the MSV picture

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To a single crossing, we associate $U^0 \subset SL_{m+n}$ where

$$\begin{aligned} U^i &= \{g \in SL_{m+n} \,|\, \dim(g(\mathbb{C}^n) \cap \mathbb{C}^m) = i\} \\ \tilde{U}^i &= \{g \in SL_{m+n}, V \subset \mathbb{C}^m | V \subset (g(\mathbb{C}^n) \cap \mathbb{C}^m), \dim V = i\} \end{aligned}$$

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Thus the complex of bimodules over $H^*_{L_{m,n}}(pt)$–$H^*_{L_{n,m}}(pt)$ (that is $R_{m,n}$–$R_{n,m}$) associated to a single crossing is

$$\to M_i[-i(i+1)] \to M_i[-i(i-1)] \to$$

where

$$M_i = I\!H^*_{L_{m,n} \times L_{n,m}}(\overline{U_i}) \cong H^*_{L_{m,n} \times L_{n,m}}(\tilde{U}_i) \cong R_{m,n-i,i} \otimes_{R_{m+n-i,i}} R_{n,m-i,i}$$

and the differential is pull and push on a correspondence over $\tilde{U}_i$ and $\tilde{U}_{i-1}$.