

# Knot invariants and higher representation theory

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*Abstract.* We describe a categorification of the tensor products of integrable representations of Kac-Moody algebras and quantum groups, based on the pictorial approach of Khovanov and Lauda.

Using structural functors of this categorification which lift the braiding and duality maps on quantum group representations, we construct knot invariants categorifying the quantum knot variants for all representations of quantum groups. We show that these invariants coincide with previous invariants defined by Khovanov for  $\mathfrak{sl}_2$  and  $\mathfrak{sl}_3$  and by Mazorchuk-Stroppel and Sussan for  $\mathfrak{sl}_n$ .

We also suggest an approach to showing that these knot homologies are functorial.

Much of the theory of quantum topology rests on the structure of monoidal categories and their use in a variety of topological constructions. In this paper, we construct a categorification of one of these: the R-matrix construction of quantum knot invariants, following Reshetikhin and Turaev [Tur88, RT90]. This theory constructs polynomial invariants of knots by assigning natural maps between tensor products of representations of a quantized universal enveloping algebra  $U_q(\mathfrak{g})$  to each ribbon tangle labeled with representations. This map is natural with respect to tangle composition, so can be reconstructed from a small number of constituents, most notably the map associated to a single ribbon twist, single crossing, single cup and single cap.

The map associated to a link whose components are labeled with a representation of  $\mathfrak{g}$  (or the corresponding highest weight) is simply a Laurent polynomial. Particular cases of these include:

- the **Jones polynomial** when  $\mathfrak{g} = \mathfrak{sl}_2$  and all strands are labeled with the defining representation.
- the **colored Jones polynomial** for other representations of  $\mathfrak{g} = \mathfrak{sl}_2$ .

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- specializations of the **HOMFLYPT polynomial** for the defining representation of  $\mathfrak{g} = \mathfrak{sl}_n$ .
- the **Kauffman polynomial** (not to be confused with the Kauffman bracket, a variant of the Jones polynomial) for the defining representation of  $\mathfrak{sp}_{2n}$ .

These special cases have been categorified to knot homologies from a number of perspectives by Khovanov and Khovanov-Rozansky [Kho00, Kho02, Kho04, Kho07, KR08a, KR07, KR08b], Stroppel and Mazorchuk-Stroppel [Str05, MSa], Sussan [Sus07], Seidel-Smith [SS06], Manolescu [Man07], Cautis-Kamnitzer [CK08a, CK08b], Mackaay, Stošić and Vaz [MSV09, MSV] and the author and Williamson [WW] but all of these have only considered minuscule representations (of which there are only finitely many in each type); in a paper still in preparation, Frenkel, Stroppel and Sussan also consider the case of the colored Jones polynomial [FSS]. However, the vast majority of representations previously had no homology theory attached to them. In this paper, we will construct such a theory for any labels; that is,

**Theorem A.** *For each  $\mathfrak{g}$ , there is a homology theory  $\mathcal{K}(L, \{\lambda_i\})$  for links  $L$  whose components are labeled by finite dimensional representations of  $\mathfrak{g}$  (here indicated by their highest weights  $\lambda_i$ ), which associates to such a link a bigraded vector space whose graded Euler characteristic is the quantum invariant of this labeled link.*

*This theory coincides with Khovanov's homologies for  $\mathfrak{g} = \mathfrak{sl}_2, \mathfrak{sl}_3$  up to grading shift when the link is labeled with the defining representation of these algebras, and the Mazorchuk-Stroppel-Sussan homology for the defining representation of  $\mathfrak{sl}_n$ .*

Conjecturally, the Mazorchuk-Stroppel-Sussan homology is canonically isomorphic to Khovanov-Rozansky homology (see [MSa, §7]).

At the moment, we have not proven that this theory is functorial, but we do have a proposal for the map associated to a cobordism. As usual in knot homology, this proposed functoriality map is constructed by picking a Morse function on the cobordism, and associating simple maps to the addition of handles, and we have no proof that this definition is independent of Morse function.

Our method for this construction is to categorify every structure on the ribbon category of  $U_q(\mathfrak{g})$ -representations used in the original definition: its braiding, ribbon structure, and rigid structure (the functor of taking duals). This approach was pioneered by Stroppel for the defining rep of  $\mathfrak{sl}_2$  [Stra, Strb] and was extended to  $\mathfrak{sl}_n$  by Sussan [Sus07] and Mazorchuk-Stroppel [MSa]. But for our approach, we must use much less familiar categories than the variations of category  $\mathcal{O}$  used by those authors and thus need to construct and check relations between functors analogous to the translation and twisting functors that appear in the  $\mathfrak{sl}_n$  case (which our construction will specialize to).

**Theorem B.** *For each ordered  $\ell$ -tuple  $\lambda = (\lambda_1, \dots, \lambda_\ell)$  of dominant weights of  $\mathfrak{g}$ , there is a graded abelian category  $\mathfrak{V}^\lambda$  such that:*

- (i) *There is a natural isomorphism from the graded Grothendieck group  $K^0(\mathfrak{V}^\lambda)$  to an integral form of the  $U_q(\mathfrak{g})$  representation  $V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_\ell}$ .*
- (ii) *We have exact functors  $\mathfrak{E}_i, \mathfrak{F}_i: D^\beta(\mathfrak{V}^\lambda) \rightarrow D^\beta(\mathfrak{V}^\lambda)$  where  $i$  ranges over nodes of the Dynkin diagram for which the induced map on  $K^0(\mathfrak{V}^\lambda)$  is the action of the Chevalley generators  $E_i, F_i$  of  $U_q(\mathfrak{g})$ . In fact, these give an action of the 2-Kac-Moody algebra as defined by Rouquier [Rou] or Khovanov-Lauda [KLc].*
- (iii) *If  $\sigma$  is a braid, then we have an exact functor  $\mathbb{B}_\sigma: D^\beta(\mathfrak{V}^\lambda) \rightarrow D^\beta(\mathfrak{V}^{\sigma(\lambda)})$  such that the induced map  $K^0(\mathfrak{V}^\lambda) \rightarrow K^0(\mathfrak{V}^{\sigma(\lambda)})$  is the action of the appropriate composition of R-matrices and flips. Furthermore, these functors induce a strong action of the braid groupoid on the categories associated to permutations of the set  $\lambda$ .*
- (iv) *If two consecutive elements of  $\lambda$  label dual representations and  $\lambda^-$  denotes the sequence with these removed, then there is a functor  $\mathbb{T}: D^\beta(\mathfrak{V}^{\lambda^-}) \rightarrow D^\beta(\mathfrak{V}^\lambda)$  which induces the quantum trace on the Grothendieck group, and similarly a functor  $\mathbb{K}: D^\beta(\mathfrak{V}^\lambda) \rightarrow D^\beta(\mathfrak{V}^{\lambda^-})$  for the coevaluation map.*
- (v) *When  $\mathfrak{g} = \mathfrak{sl}_n$ , our category is equivalent to a subcategory of parabolic category  $\mathcal{O}$  for  $\mathfrak{gl}_N$  (for some  $N$ , different from  $n$ ), and our structure functors can be described in terms of translation, twisting, etc. functors on  $\mathcal{O}$ .*

The quantum knot invariants are given by a composition of the decategorifications of the functors constructed in Theorem B, as described in [CP95, §4]; combining the functors themselves in the same pattern gives the knot homology of Theorem A.

We see no reason to think that our category has a similar description in terms of classical representation theory when  $\mathfrak{g} \not\cong \mathfrak{sl}_n$ , though we would be quite pleased to be proven wrong in this speculation.

We would also like to point out that the first three parts of this theorem depend in no way on  $\mathfrak{g}$  being finite dimensional, as opposed to symmetrizable Kac-Moody. In fact, we show in Section 4 that the categories associated to  $\mathfrak{g} = \widehat{\mathfrak{sl}}_n$  are subcategories of representations of cyclotomic  $q$ -Schur algebras for  $q$  an  $n$ th root of unity, as one would expect based on analogy with the result for  $\mathfrak{g} = \mathfrak{sl}_n$ , and the work of Brundan and Kleshchev [BK09].

This construction can be seen as part of the program of “higher representation theory,” in the sense of Rouquier. We propose that the category  $\mathfrak{V}^\lambda$  mentioned above should be considered as a “tensor product category” of irreducible 2-representations. This is not the tensor product of categories in the naive sense, but rather an extension of this naive tensor product by additional objects and morphisms.

We believe our category will be derived equivalent to that defined by Hao Zheng in [Zheb], and that, in fact, this will prove to be a simple consequence of the work of Vasserot-Varagnolo [VV]. Rouquier has proposed a more abstract notion of the tensor product of 2-representations [Roua], which has yet to appear in print, but which we also hope will agree with our concrete one on irreducible representations.

While we do not develop this direction in this paper because it is not necessary for the definition of our knot invariants, the constructions in this paper were strongly inspired by the geometry of quiver varieties, and thus owe a great debt to the works of Lusztig, Ginzburg and Nakajima which may not be immediately obvious. In fact, the same algebra used to construct the categories in Theorem B was independently defined by Zheng in an unpublished manuscript [Zhea] based on geometric considerations. We believe that our constructions will ultimately prove to be a special case of a uniform construction of a geometric category  $\mathcal{O}$  which will be defined in forthcoming work of the author with Braden, Licata and Proudfoot [BLPWb] and will coincide with Zheng's categories [Zheb] for quiver varieties after applying a derived equivalence.

In particular, we expect that the categories defined in this paper will be related to not just the geometry of quiver varieties, but to that of affine Grassmannians, especially to the slices relating pairs of comparable orbits. This is suggested by the symplectic duality conjectures of the author and same collaborators as above [BLPWb].

Let us now summarize the structure of the paper.

- In Section 1, we define the categories that appear in Theorem B. These are defined as the representation categories of certain finite dimensional algebras  $E^\lambda$  defined via a graphical calculus reminiscent of Khovanov-Lauda. This graphical calculus gives an easy description of their module structure over the functors of Theorem B, part (ii).

We also develop some results on their structure, the most important being a special class of modules which we term **standard modules**. These are typically not the standard modules of a quasi-hereditary structure, but rather of a weaker standardly stratified structure.

- In Section 2, we prove Theorem B except part (v). That is, we construct the functors lifting the ribbon, braided and rigid structure of the monoidal category of  $U_q(\mathfrak{g})$ -representations. Each of these functors is given by tensor product or Hom with explicitly presented bimodules very naturally suggested by the graphical calculus.
- In Section 3, we prove Theorem A using the maps constructed in Theorem B and a small number of explicit computations. We also suggest a map for the functoriality along a cobordism between links; however, this map is defined by choosing a handle decomposition of the cobordism, and at the moment, we have no proof that the induced map is independent of the handle decomposition.
- In Section 4, we consider the case  $\mathfrak{g} = \mathfrak{sl}_n$ . In this case, we employ results of Brundan and Kleshchev to show that  $\mathfrak{A}^\lambda$  is in fact a subcategory of a category  $\mathcal{O}$ , and that all the functors appearing Theorem B are previous defined

functors on these categories. This allows us to show the portions of Theorem A and B regarding comparisons to Khovanov homology and Mazorchuk-Stroppel-Sussan homology.

**Notation.** We let  $\mathfrak{g}$  be a finite-dimensional simple complex Lie algebra, which we will assume is fixed for the remainder of the paper. Consider the weight lattice  $Y(\mathfrak{g})$  and root lattice  $X(\mathfrak{g})$ . Thus we have simple roots  $\alpha_i$  and coroots  $\alpha_i^\vee$ . Let  $c_{ij} = \alpha_j^\vee(\alpha_i)$  be the entries of the Cartan matrix. We let  $\langle -, - \rangle$  denote the symmetrized inner product on  $Y(\mathfrak{g})$ , fixed by the fact that the shortest root has length  $\sqrt{2}$  and

$$2 \frac{\langle \alpha_i, \lambda \rangle}{\langle \alpha_i, \alpha_i \rangle} = \alpha_i^\vee(\lambda).$$

As usual, we let  $2d_i = \langle \alpha_i, \alpha_i \rangle$ , and for  $\lambda \in Y(\mathfrak{g})$ , we let

$$\lambda^i = \alpha_i^\vee(\lambda) = \langle \alpha_i, \lambda \rangle / d_i.$$

We let  $\rho$  be the unique weight such that  $\alpha_i^\vee(\rho) = 1$  and  $\rho^\vee$  the unique coweight such that  $\rho^\vee(\alpha_i) = 1$ . We note that since  $\rho \in 1/2X$  and  $\rho^\vee \in 1/2Y^*$ , for any weight  $\lambda$ , the numbers  $\langle \lambda, \rho \rangle$  and  $\rho^\vee(\lambda)$  are not necessarily integers. However,  $2\langle \lambda, \rho \rangle$  and  $2\rho^\vee(\lambda)$  are (not necessarily even) integers.

Throughout the paper, we will use  $\lambda = (\lambda_1, \dots, \lambda_\ell)$  to denote an ordered  $\ell$ -tuple of dominant weights, and always denote  $\lambda = \sum_i \lambda_i$ .

We let  $U_q(\mathfrak{g})$  denote the deformed universal enveloping algebra of  $\mathfrak{g}$ ; that is, the associative  $\mathbb{k}(q)$ -algebra given by generators  $E_i, F_i, K_\mu$  for  $i$  and  $\mu \in Y(\mathfrak{g})$ , subject to the relations:

- i)  $K_0 = 1, K_\mu K_{\mu'} = K_{\mu+\mu'}$  for all  $\mu, \mu' \in Y(\mathfrak{g})$ ,
- ii)  $K_\mu E_i = q^{\alpha_i^\vee(\mu)} E_i K_\mu$  for all  $\mu \in Y(\mathfrak{g})$ ,
- iii)  $K_\mu F_i = q^{\alpha_i^\vee(\mu)} F_i K_\mu$  for all  $\mu \in Y(\mathfrak{g})$ ,
- iv)  $E_i F_j - F_j E_i = \delta_{ij} \frac{\tilde{K}_i - \tilde{K}_{-i}}{q^{d_i} - q^{-d_i}}$ , where  $\tilde{K}_{\pm i} = K_{\pm d_i \alpha_i}$ ,
- v) For all  $i \neq j$

$$\sum_{a+b=-c_{ij}+1} (-1)^a E_i^{(a)} E_j E_i^{(b)} = 0 \quad \text{and} \quad \sum_{a+b=-c_{ij}+1} (-1)^a F_i^{(a)} F_j F_i^{(b)} = 0.$$

This is a Hopf algebra with coproduct on Chevalley generators given by

$$\Delta(E_i) = E_i \otimes 1 + \tilde{K}_i \otimes E_i \quad \Delta(F_i) = F_i \otimes \tilde{K}_{-i} + \tilde{1} \otimes F_i$$

We will always use the Lusztig (divided powers) integral form generated over  $\mathbb{Z}[q, q^{-1}]$  by  $\frac{E_i^n}{[n]_q!}, \frac{F_i^n}{[n]_q!}$  for all integers  $n$  of this quantum group. The integral form of the representation of highest weight  $\lambda$  over this quantum group will be denoted by  $V_\lambda$ , and  $V_\lambda = V_{\lambda_1} \otimes \dots \otimes V_{\lambda_\ell}$ .

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1. THE ALGEBRAS  $E^\lambda$

Let us proceed to the construction of the categories of Theorem B as modules over certain algebras.

**1.1. Definition and basic properties.** As we described, the definition of our algebra is pictorial, and similar in flavor to the approach of Khovanov and Lauda [Lau, KL<sub>a</sub>, KL<sub>b</sub>, KL<sub>c</sub>].

The generators of our algebra are pictures in  $\mathbb{R}^2$  consisting of red and black oriented smooth curves decorated with a number (possibly 0) of dots such that:

- each curve begins on the line  $y = 0$  and ends on the line  $y = 1$
- each curve is never tangent to a horizontal line
- locally around each point, our diagram is either a single line or one of the pictures:



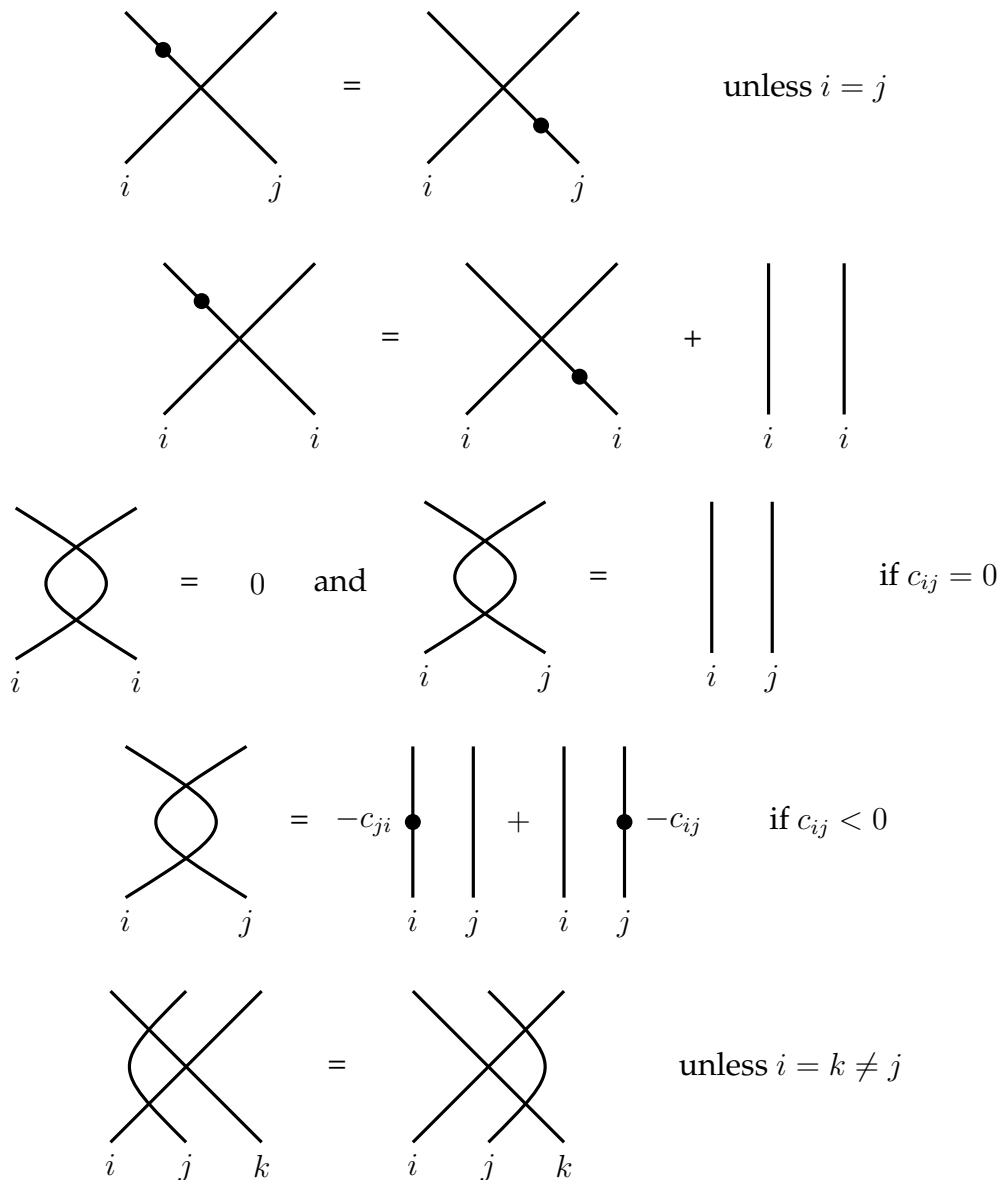
In particular, red lines are never allowed to cross, and no pair of lines are allowed to meet the lines  $y = 0$  or  $y = 1$  in the same point.

We will only ever be interested in these pictures up to isotopy.

Consider the algebra over a field  $\mathbb{k}$  of characteristic 0 whose generators are pictures as above, with each black line labeled by a simple root of  $\mathfrak{g}$ , and each red line labeled with a dominant weight. Multiplication is given by the stacking of diagrams if the

pattern of red and black lines with their labels can be isotoped to match up at  $y = 1$  in the first diagram and  $y = 0$  in the second and is defined to be 0 otherwise. Of course, this stacking must be followed by smoothing any kinks at the joins of the lines (which is unique up to isotopy) and vertical scaling to match the ends up with the correct horizontal lines. By convention the product  $ab$  means stacking the diagram  $b$  on top of the diagram  $a$ .

We will also require some relations; all but one of these will be “local” in nature, so that by equating two diagrams, we really mean that we identify any two diagrams which are identical outside a small circle, and which differ by the relation inside that circle. The relations between black lines are those of Khovanov and Lauda [KLb], which we reproduce below.





**Remark 1.** *The exact form of these relations (in particular the question of signs) is a tricky question about which there seems to be some confusion. The signs will not be important for any of our calculations, so it seems that our theorems hold in any set of relations where the cyclotomic quotient has the correct Grothendieck group. Since previous work of the author and Stroppel [SW] (and less directly with Braden, Licata and Proudfoot [BLPWa]) suggests the relations of  $R(\nu)$  in [KLb] is the most geometrically natural, these are the relations we will use for the time being.*

*Where it seems most likely that signs will appear is in determining functoriality of these knot homologies with respect to cobordisms between knots. The original Khovanov homology had serious sign problems (see Jacobsson [Jac04]) which were fixed by Clark, Morrison and Walker [CMW09] by making a modification which was not visible just on the level of vector spaces. It may be that there is only one set of relations for which functoriality works, as should be teased out of careful analysis of movie moves of Carter and Saito [CS93].*

**Grading.** This algebra is graded with degrees given by

- a black/black crossing:  $-\langle \alpha_i, \alpha_j \rangle$ ,
- a black dot:  $\langle \alpha_i, \alpha_i \rangle = 2d_i$
- a red/black crossing:  $\langle \alpha_i, \lambda \rangle = d_i \lambda^i$ .

**Duality.** This algebra is endowed with a natural anti-automorphism  $a \mapsto \dot{a}$  given by reflecting graphs in the horizontal plane. If  $M$  is a right module over this algebra, we let  $\dot{M}$  be the left module given by twisting the action by this anti-automorphism. In particular, if  $M$  is a finite-dimensional right module, then we can define a dual right module by  $M^* = \dot{M}^*$ , since both vector space dual and the anti-automorphism interchange left and right modules.

**Definition 1.1.** *For a sequence of weights  $\lambda = (\lambda_1, \dots, \lambda_\ell)$ , we let  $E^\lambda$  be the subalgebra of the universal one defined above where the red lines, in order, are labeled with the elements of  $\lambda$ . We let  $\mathfrak{A}^\lambda = E^\lambda - \text{mod}$  be the category of finite dimensional representations of  $E^\lambda$ , and let  $\mathcal{V}^\lambda = D^\beta(\mathfrak{A}^\lambda)$  be its derived category given by complexes which are bounded below, such that the total cohomology of any object is finite dimensional.*

*We let  $E_\alpha^\lambda$  for  $\alpha \in Y(\mathfrak{g})$  be the subalgebra of  $E^\lambda$  where the sum of the roots associated to the black strands is  $\sum_i \lambda_i - \alpha$ , and  $\mathfrak{A}_\alpha^\lambda, \mathcal{V}_\alpha^\lambda$  be the corresponding categories.*

We use the bounded-below derived category  $D^\beta(\mathfrak{A}^\lambda)$  rather than the bounded derived category  $D^b(\mathfrak{A}^\lambda)$  since these algebras may not have finite global dimension. We require the total cohomology to be finite dimensional so that the Grothendieck groups of the derived and abelian categories obviously coincide.

Note that  $E^\lambda = \bigoplus_\alpha E_\alpha^\lambda$  and  $\mathcal{V}^\lambda = \bigoplus_\alpha \mathcal{V}_\alpha^\lambda$ . As suggested by the notation,  $\mathfrak{A}^\lambda$  is the category which appears in Theorem B.

**Remark 2.** *While the abelian category  $\mathfrak{A}^\lambda$  is a more comfortable object for most readers, we simply cannot avoid using the derived category  $\mathcal{V}^\lambda$  for the definition of our functors. The*

Grothendieck groups  $K^0(\mathfrak{B}^\lambda)$  and  $K^0(\mathcal{V}^\lambda)$  are canonically isomorphic, so in a certain sense we can think of both these categories as categorifying the representations of  $U_q(\mathfrak{g})$ . However, our link invariant can only be defined on using derived categories, and should really be thought of as valued in the derived category of graded vector spaces.

For a sequence of simple roots  $\mathbf{i} = (i_1, \dots, i_n)$ , and a weakly increasing map  $\kappa : [1, \ell] \rightarrow [0, n]$ , we can define a projective object in  $P_1^\kappa$  of  $\mathfrak{B}^\lambda$  by considering the pictures whose bottom ends are labeled by the roots in the order given by  $\mathbf{i}$ , with the  $j$ th red line immediately right of the  $\kappa(j)$ th black line and if  $\kappa(j)$ 's agree, the order is preserved. We can generalize this notion a bit by allowing multiplicities  $\vartheta_j$ ; we associate a projective to the sequence  $(i_1^{(\vartheta_1)}, \dots, i_n^{(\vartheta_n)})$  which is a submodule of the projective for the sequence where  $i_j^{(\vartheta_j)}$  has been expanded to  $\vartheta_j$  instances of  $i_j$ . This is the projective given by multiplying each block of strands in the expanded projective on the bottom by the idempotent denoted  $e_{\vartheta_j}$  in [KLb, §2].

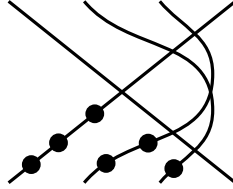


FIGURE 1. The idempotent  $e_4$ .

For each choice of a reduced word  $\mathbf{w}$  for a permutation of  $n + \ell$  letters, we have an element  $\psi_{\mathbf{w}}$  of  $P_1^\kappa$  given by replacing the simple reflection  $(i, i + 1)$  with the crossing of the  $i$  and  $i + 1$ st strands (red or black) and multiplying out the result.

**Proposition 1.2.** *For any fixed choice of reduced word for each permutation, the elements  $\psi_{\mathbf{w}}$  generate  $P_1^\kappa$  as a module over the subalgebra generated by the  $y_i$ 's.*

*Proof.* All we need show is that  $\psi_{\mathbf{w}}$  can be rewritten in terms of  $y_i$ 's times  $\psi_{\mathbf{w}'}$  for our fixed choice of reduced words and shorter diagrams.

If  $\mathbf{w}$  is not a reduced word in the symmetric group, then by applying braid relations (which hold modulo shorter words), we can assume that there are two consecutive crossings of the same strands, which can be simplified using the relations.

If  $\mathbf{w}$  is a reduced word, then the fixed reduced word corresponding to the same permutation  $\mathbf{w}'$  differs from  $\mathbf{w}$  by Tits moves, so the difference between  $\psi_{\mathbf{w}} - \psi_{\mathbf{w}'}$  can thus be written in terms of shorter diagrams. □

**1.2. Relationship to QHA's.** Let  $R$  be the universal QHA; that is the quotient of the set of pictures as described earlier only using black strands by the Khovanov-Lauda relations. This algebra has a block decomposition according to the sum of the labels of the black strands. Let  $R(\nu)$  be the subalgebra where the labels add to  $\nu$  (this notation coincides with that of [KLb]).

**Definition 1.3.** *The cyclotomic QHA for a weight  $\lambda$  is the quotient of  $R(\nu)$  by the 2-sided ideal generated by the elements  $x_1^{(\lambda, \alpha_{i_1})} e(\mathbf{i})$ .*

If  $\lambda = (\lambda)$ , then we will simplify notation by writing  $E^\lambda$  for  $E^\lambda$ .

**Proposition 1.4.** *The cyclotomic QHA for  $\lambda$  is isomorphic to  $E^\lambda$ .*

*Proof.* By relations (1.1) and (1.2) above, we have a homomorphism from the cyclotomic QHA to  $E^\lambda$ , sending K-L generators to K-L generators.

On the other hand, to find any element in the kernel of this map, we would have to apply one of the relations above, which would necessitate having a black strand to the left of the single red one. Thus, that relation would be implied by the cyclotomic one.  $\square$

The cyclotomic quotient is important because it categorifies the Lusztig integral form of the simple  $U_q(\mathfrak{g})$ -representation of highest weight  $\lambda$  (which we denote  $V_\lambda$ ), as was recently shown by Lauda and Vazirani.

**Proposition 1.5** ([LV, Theorem 7.8]). *There is a isomorphism of  $U_q^+(\mathfrak{g})$ -representations  $K^0(\mathfrak{A}^\lambda) \cong V_\lambda$ .*

**1.3. The module category structure.** Let  $\mathcal{V}^\infty = D^{per}(R - \text{mod})$  be the derived category of bounded perfect complexes of representations of the QHA  $R$ ; this is the categorification of the lower triangular part of the quantum group defined by Rouquier and Khovanov-Lauda. In the work of Khovanov-Lauda, this category was typically divided into blocks which are the representations of the algebra  $R(\nu)$ , but it is more convenient for us to consider them all simultaneously as one category. This category has a monoidal structure, which corresponds to the product on the quantum group, as shown in [KLb]. We now show how  $\mathcal{V}^\lambda$  is a module category over  $\mathcal{V}^\infty$ .

As in the work of Khovanov and Lauda, the action functor is given by an induction type functor. We have a natural map  $E^\lambda \otimes R \rightarrow E^\lambda$  given by ‘‘horizontal composition,’’ that is, taking the disjoint union of the two diagrams in  $\mathbb{R}^2$ , with the diagram from  $R$  on the right.

**Theorem 1.6.** *The functor  $- \otimes_{E^\lambda \otimes R}^L E^\lambda : \mathcal{V}^\lambda \otimes \mathcal{V}^\infty \rightarrow \mathcal{V}^\lambda$  makes  $\mathcal{V}^\lambda$  into a  $\mathcal{V}^\infty$  module category.*

*Proof.* This follows immediately from the fact horizontal composition is associative. We defer the proof that this preserves the condition of finite total cohomology to Section 1.4.  $\square$

In particular, we have a natural projective module  $R(\alpha_i)$  of  $R$  corresponding to each simple root.

**Definition 1.7.** *We denote the functor of  $M \mapsto M \otimes^L R(\alpha_i)$  by  $\mathfrak{F}_i$ . Since the functors of action of an  $R$ -module correspond to the action of Hecke correspondences on quiver varieties,*

we will refer to them as **Hecke functors**. By the definition above, this is simply extension of scalars for the map  $\nu_i : E^\lambda \rightarrow E^\lambda$  given by horizontal composition with a single vertical line.

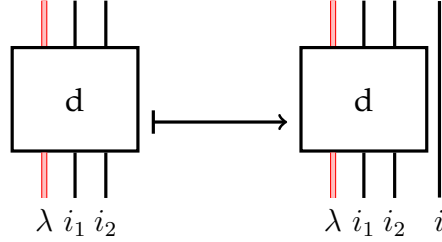


FIGURE 2. The map  $\nu_i$ .

Consider the right adjoint  $\mathfrak{F}_i^R : \mathfrak{Y}^\lambda \rightarrow \mathfrak{Y}^\lambda$  to the functor  $\mathfrak{F}_i$ ; this is a restriction of scalars and thus exact. Let  $\mathfrak{E}_i$  be the functor given by the direct sum of  $\mathfrak{F}_i^R(\langle \alpha_i, \lambda - \alpha \rangle - d_i) : \mathfrak{Y}_\alpha^\lambda \rightarrow \mathfrak{Y}_{\alpha+\alpha_i}^\lambda$ .

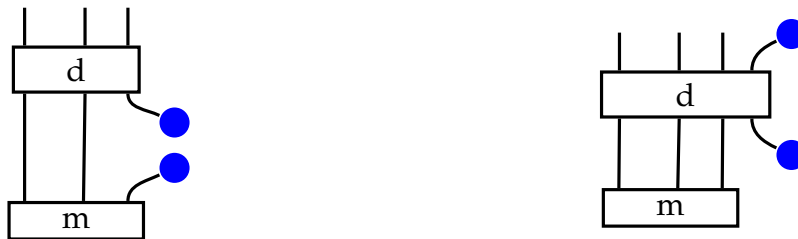
**Proposition 1.8.** *These functors give a weak categorification of the universal enveloping algebra; in the notation of Rouquier, this would be an action of the 2-category  $\mathfrak{A}$  associated to  $\mathfrak{g}$ .*

*Proof.* We have already shown that  $\mathfrak{F}_i$  give an action of the categorification of the lower half of  $U_q^+(\mathfrak{g})$ . Thus, we need only show that the maps inverted by Rouquier in [Rouib, §4.1.3] are in fact isomorphisms. Let us recall his prescription. We note that our conventions have reversed  $E_i$  and  $F_i$  from Rouquier's, so our formulas differ from his by a Cartan involution.

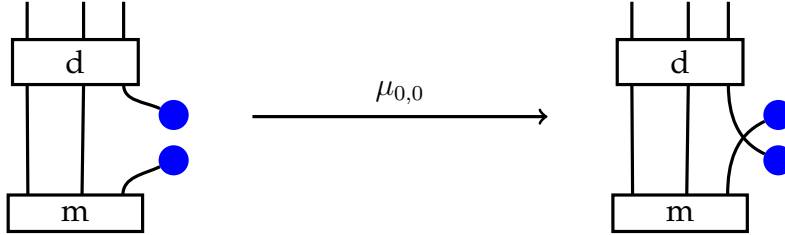
For any integer  $n$ , we let  $n^+ = \max(n, 0)$  and  $n^- = \max(-n, 0)$ . We can assume  $M$  to lie in the block  $\mathfrak{Y}_\nu^\lambda$  for fixed  $\nu \in Y(\mathfrak{g})$ , and will do so for the rest of the proof. Rouquier defines a natural transformation

$$\mu : \mathfrak{F}_i \circ \mathfrak{E}_i \oplus \text{Id}^{\alpha_i^\vee(\nu)^-} \rightarrow \mathfrak{E}_i \circ \mathfrak{F}_i \oplus \text{Id}^{\alpha_i^\vee(\nu)^+}.$$

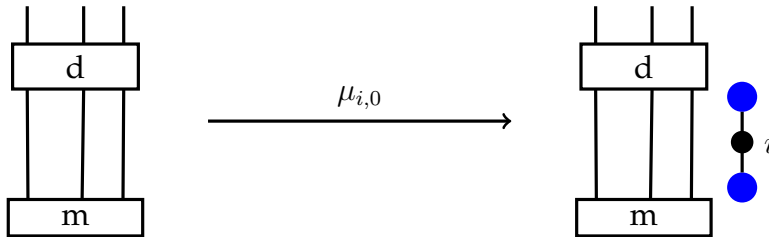
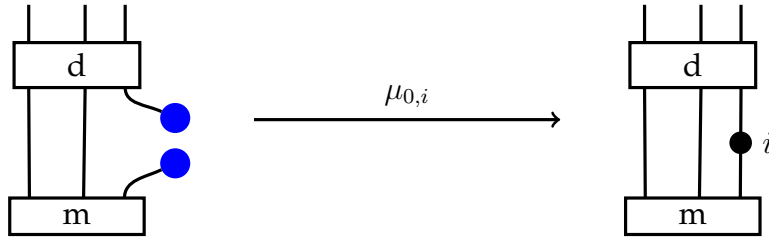
It will be easiest for us to represent this map graphically. We can represent elements of  $\mathfrak{E}_i \circ \mathfrak{F}_i(M)$  or  $\mathfrak{F}_i \circ \mathfrak{E}_i(M)$  for some module of  $M$  by pictures



The first component of the map  $\mu$  is the map that pulls the two blue dots past each other, creating a crossing of the black strands.



The others are defined by



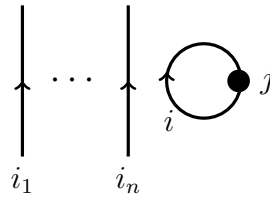
First, let us show injectivity of the map

$$\mu : \mathfrak{F}_i \circ \mathfrak{E}_i \oplus \text{Id}^{\alpha_i^\vee(\nu)^-}(M) \rightarrow \mathfrak{E}_i \circ \mathfrak{F}_i \oplus \text{Id}^{\alpha_i^\vee(\nu)^+}(M)$$

whose components are defined above for a fixed module  $M$ .

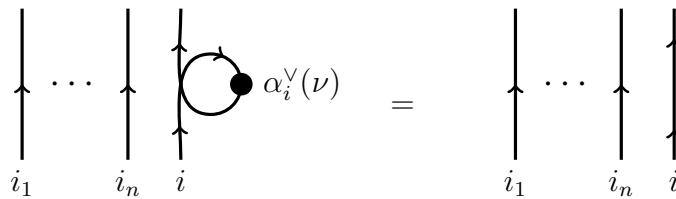
Our approach is that the formulas for constructing an inverse to this map are encoded in the graphical calculus used by Khovanov and Lauda for constructing a categorification of the entire quantum group. Within the confines of this proof, we will assume some familiarity on the part of the reader with the constructions of [KLC]. The value of this graphical calculus is that it allows to write very compactly certain formulas which make sense without using their more complicated graphical calculus, but would be extremely laborious to write explicitly.

For example, the bubble slide [KLC, Proposition 3.4] shows that



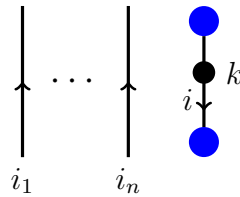
is a shorthand for an expression which is 0 if the degree of the bubble is negative,  $e_{i,\kappa}$  if it is degree 0, and a complicated linear combination of symmetric polynomials in the dots times  $e_{i,\kappa}$  if the degree is positive.

Similarly, we have an equality

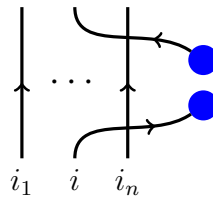


and the diagram on the left is zero if the number of dots on the loop-de-loop is less than  $\alpha_i^\nu$ .

Finally, the diagram



is a shorthand for a diagram obtained using Khovanov and Lauda's relation [KLC, (3.6)] to pull the downward oriented strand to the left. This results in a diagram where the downward oriented strand passes one step further to the left, plus correction terms of the form:



(we have drawn this schematically and left off the dots which will appear in the proper expression for the correction terms) which can all be interpreted as elements of  $\mathfrak{F}_i \circ \mathfrak{E}_i$ . Finally, when the downward pointing strand reaches the far left of the diagram, that term is declared to be 0, and only the correction terms remain.

There is a map  $\chi_j : \mathfrak{E}_i \circ \mathfrak{F}_i \rightarrow \text{Id}$  given by closing the ends with the addition of  $j$  dots and evaluating by Khovanov and Lauda's graphical calculus using relations [KLC, (3.7), Prop. 3.3]. We add the additional relation that any positive degree bubble on the far left of the diagram is 0; note that one cannot apply this at other places in the diagram, since one must take into account the correction terms that appear in bubble slides. We note now, and will use freely for the rest of the proof, that a 0 degree bubble can simply be "popped" and removed from the diagram without changing the value of the diagram.

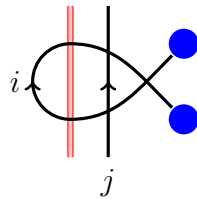
If  $\alpha_i^\vee(\nu) \geq 0$ , then  $\chi_j$  kills the image of  $\mu_{0,0}$  (since  $j \leq \alpha_i^\vee(\nu)$ ), and  $\chi_{\alpha_i^\vee(\nu)+-j} \circ \mu_{j,0} \cong \text{Id}$ . Thus, if any component of an element  $e$  in any of the copies of  $\text{Id}$  is non-trivial, it is not in the kernel. Thus, we must now show that  $\mu_{0,*}$  is injective.

Consider the map  $\zeta_{0,*} : \mathfrak{E}_i \circ \mathfrak{F}_i \oplus \text{Id}^{\alpha_i^\vee(\nu)^-} \rightarrow \mathfrak{F}_i \circ \mathfrak{E}_i$  given by constructing the diagram (the open blue nodes represent the position of the blue nodes of the diagram the map is applied to)

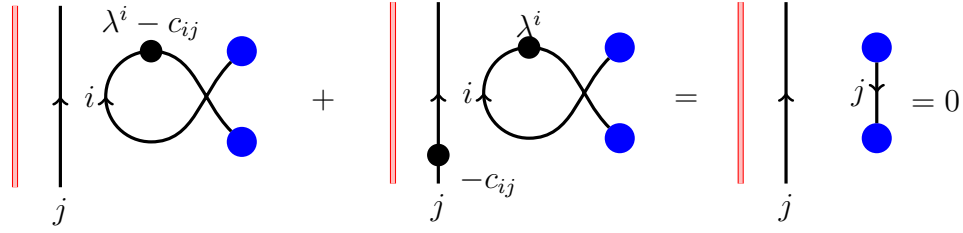
$$(1.3) \quad \begin{array}{c} \uparrow \uparrow \uparrow \\ \boxed{d} \\ \uparrow \uparrow \uparrow \\ \boxed{m} \end{array} \begin{array}{c} \circ \\ \circ \end{array} \begin{array}{c} \bullet \\ \bullet \end{array} + \sum_{i+j+k=-2} \begin{array}{c} \uparrow \uparrow \uparrow \\ \boxed{m_i} \end{array} \begin{array}{c} \circ \\ \bullet \end{array} \begin{array}{c} \bullet \\ \bullet \end{array} \begin{array}{c} \bullet \\ \bullet \end{array}$$

and then applying the relations from [KLC] to obtain a diagram which is in  $\mathfrak{F}_i \circ \mathfrak{E}_i$ , as described before.

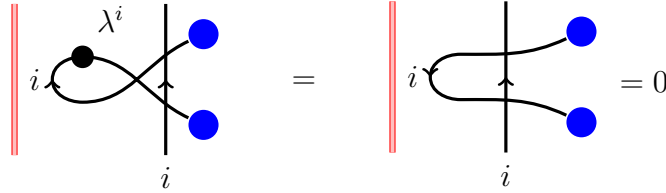
For those readers suspicious of "proof by picture," the linchpin of the argument is the fact that this is a *bona fide* map of  $E^\lambda$  representations. In particular, why the evaluation procedure we wrote earlier sends diagrams with a black strand at the far left of the diagram to diagrams of the same type. By pushing crossings right (which never disturbs having a strand on the far right), we can see that the only case where the resolution procedure above will take us out of the ideal of elements with black strands at the far left is



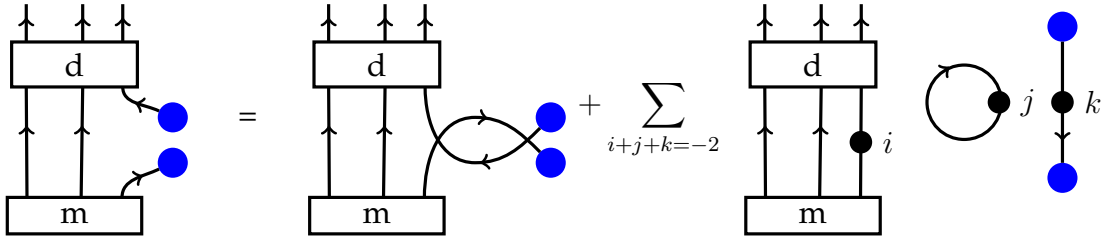
If  $j \neq i$ , then the picture above is equal to



On the other hand, if  $i = j$ , then we are in the situation of  $\mathfrak{sl}_2$ , which has been treated in a number of places, most notably by Rouquier [Rouq, §4], following work of Rouquier and Chuang [CR] and by Lauda [Lau] in the graphical calculus we follow. While the consistency we require follows from the work of these authors (for example, by [Lau, 8.3]), for completeness, note that we can evaluate the picture above by pulling the vertical black strand across the black crossing using the relation of [KLC, Proposition 3.5]. This shows that modulo elements of  $\mathfrak{F}_i \circ \mathfrak{E}_i$  which pass left of the red line, the picture above is congruent to

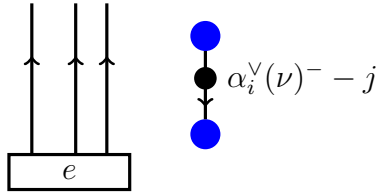


On the other hand, by [KLC, (3.5)],  $\zeta_{0,*}$  splits  $\mu_{0,*}$ , since



We note that this expression is tautological based on the interpretation we have given these diagrams. The point is that simply uncrossing the strands is not well defined a map, where the formula given in (1.3) is. Therefore,  $\mu_{0,*}$  and thus  $\mu$  is injective.

On the other hand, we now show that  $\mu$  is surjective: assume that  $e \in \mathfrak{E}_i \circ \mathfrak{F}_i(M) \oplus \text{Id}^{\alpha_i^\vee(\nu)^-}$  is not in the image of  $\mu$ , and let  $j$  is the smallest index for which the component in that copy of  $\text{Id}(M) = M$  is non-trivial. Consider the element of  $e' \in \mathfrak{F}_i \circ \mathfrak{E}_i(M)$  defined by



In order to interpret this as an element of  $\mathfrak{F}_i \circ \mathfrak{E}_i(M)$ , one must do the same algorithm as before, pulling the downward oriented strand to the left until it reaches the far left and is killed, leaving us only with correction terms.

The image of  $\mu_{0,j}$  applied to this element is exactly the projection of  $e$  to the  $j$ th copy of  $\text{Id}$  (since closing with the addition of  $j$  dots just creates a bubble of degree zero, which can be “popped”) and its projection to the  $k$ th copy for all  $k$  lower than  $j$  is trivial (since this contains a bubble of negative degree). Thus  $e - \mu(e)$  is not in the image of  $\mu$ , and its first non-zero component is in a higher indexed component of  $\text{Id}$ . By induction, we may thus assume  $e \in \mathfrak{E}_i \circ \mathfrak{F}_i$ .

If an element of  $\mathfrak{E}_i \circ \mathfrak{F}_i(M)$  is not in the image of  $\mu_{*,0}$ , then it may be assumed to be a sum of diagrams where the blue dots are directly connected, and that connection is labeled with more than  $\alpha_i^\vee(\nu)$ -dots. On the other hand, any diagram of that form be written as a sum of one where the black strand between the blue dots is pulled all the way to the left of the diagram (which is thus 0), plus correction terms which are either in the image of  $\mu$ , or involve strictly fewer dots on the strand between blue dots. By induction,  $\mu$  must thus be surjective.

Thus  $\mu$  is an isomorphism. □

Recall that we call an algebra  $A$  **symmetric** if  $A$  and  $A^*$  are isomorphic as  $A$ -bimodules. This isomorphism can also be interpreted as a non-degenerate symmetric trace on  $A$ .

**Proposition 1.9.** *The functors  $\mathfrak{F}_i$  and  $\mathfrak{E}_i$  are biadjoint. In particular, the cyclotomic QHA  $E_\alpha^\lambda$  is symmetric, with trace of degree  $\langle \alpha, \alpha \rangle - \langle \lambda, \lambda \rangle$ .*

*Proof.* This is an immediate consequence of [Roub, Theorem 5.16]. The only thing we need to show is that the action of  $\mathfrak{F}_i$  and  $\mathfrak{E}_i$  on  $\mathcal{V}^\lambda$  is integrable, that is, that any object is killed by a large enough power of either representation. This is clear for  $\mathfrak{E}_i$ , since each application of  $\mathfrak{E}_i$  decreases the number of strands. For  $\mathfrak{F}_i$ , one can show this directly, but it is simpler to appeal to Theorem 1.22, which shows that applying  $\mathfrak{F}_i$  a sufficiently large number of times will send a module to a category corresponding to a trivial weight space, and thus 0.

The algebra  $E^\lambda$  is a direct sum of projectives of the form  $\mathfrak{F}_i P_\emptyset$ . The dual to this projective is  $(\mathfrak{F}_i^R)^R P_\emptyset \cong \mathfrak{F}_i P_\emptyset$  which is isomorphic to  $\mathfrak{F}_i P_\emptyset$  by the biadjunction. These isomorphisms combine to give an isomorphism  $E^\lambda \cong (E^\lambda)^*$ . □

In fact, Rouquier gives an explicit trace, which we present here reformulated in pictorial language.

Given a diagram  $d$  which represents an element of  $R(\nu)$ , such that the labels at the top and bottom match, we can close this diagram on the right as one would a braid to obtain a knot, as shown in Figure 3. We can interpret this as a 2-morphism from  $\text{id}_\lambda$  to itself in the categorification of the entire quantum group given by Khovanov-Lauda in [KLc] (pictorially, we label the unbounded component outside our diagram with  $\lambda$ ). By [KLc, Proposition 3.6], the endomorphism algebra of  $\text{id}_\lambda$  is positively graded over the base field, so it makes sense to speak of the constant term of such a morphism.

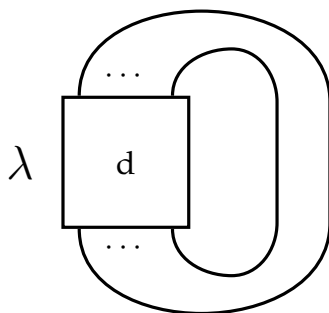


FIGURE 3. Closing a diagram

**Proposition 1.10** ([Roub, §4.1.4 & Theorem 5.16]). *The Frobenius trace  $\tau_\lambda : E^\lambda \rightarrow \mathbb{k}$  is defined by*

- If  $d$  has matching top and bottom labels,  $\tau_\lambda(d)$  is the constant term of its closure, defined above.
- If  $d$  does not then  $\tau_\lambda(d) = 0$ .

**1.4. Standard modules.** When analyzing the structure of representation-theoretic categories, such as the categories  $\mathcal{O}$  appearing in Stroppel’s construction of Khovanov homology [Stra], a crucial role is played by the parabolic Verma modules. The property of “having objects like Verma modules” was formalized by Cline-Parshall-Scott as the property of being **quasi-hereditary** [CPS88]. Unfortunately, this is too strong of an assumption for us; as we noted earlier, the cyclotomic QHA is conjectured to be Frobenius, and thus very far from being quasi-hereditary.

Luckily, our categories satisfy a weaker condition: they are **standardly stratified**, as defined by the same authors [CPS96]. To show this, we must construct a collection of modules which are called **standard**, and show that projectives have a filtration by these modules compatible with a pre-order.

We define a preorder on  $(\mathbf{i}, \kappa)$ 's by calling  $(\mathbf{i}, \kappa) \leq (\mathbf{i}', \kappa')$  if

$$\sum_{k \leq \kappa(j)} \alpha_{i_k} \leq \sum_{k \leq \kappa'(j)} \alpha_{i'_k} \quad \text{for all } j \in [1, \ell].$$

We can extend this to a preorder on simples by  $L \leq L'$  if  $L$  is a quotient of  $S_{\mathbf{i}}^{\kappa}$  and  $L'$  of  $S_{\mathbf{i}'}^{\kappa'}$  and  $(\mathbf{i}, \kappa) \leq (\mathbf{i}', \kappa')$ . Note that this preorder is entirely insensitive to permutations of the black strands which do not cross any red strands.

**Definition 1.11.** *By convention, we call a red/black crossing where black strands go from NW to SE **left** and the mirror image of such a crossing **right**.*

*Note that this terminology does not apply to black/black crossings; if we call a crossing left or right we are implicitly assuming it is black/red.*



a left crossing



a right crossing

The point of these definitions is that the maps induced between projectives by adding a left crossing on the bottom always sends a projective to one smaller in this preorder, and *vice versa* for right crossings.

Let  $L \subset P_{\mathbf{i}}^{\kappa}$  be the submodule generated by diagrams with no right crossings, and at least one left crossing.

We note that we can choose representatives of all permutations where all left crossings occur before all black or right ones. By Proposition 1.2, when any element is written in terms of the diagrams for these reduced words (times arbitrary polynomials in the dots),  $L$  is the span of all diagrams with any left crossings. In particular,

**Proposition 1.12.** *The image of any map from a projective higher than  $(\mathbf{i}, \kappa)$  in the preorder  $\leq$  is contained in  $L \subset P_{\mathbf{i}}^{\kappa}$ , and these images generate  $L$ . That is, the submodule  $L$  is the “trace” of these projectives.*

This is the definition of standard modules given in (for instance) [MSb], so our terminology matches theirs.

*Proof.* Generation is clear: any diagram with only left crossings defines a map from a higher projective to  $P_{\mathbf{i}}^{\kappa}$  with the image of the idempotent being the original diagram.

To show that any such image lands in  $L$ , consider an arbitrary map from a higher projective. This is given by a sum of diagrams in  $P_{\mathbf{i}}^{\kappa}$  whose upper end points are given by the idempotent for that projective, so we can write these in terms of the representatives of all permutations where all left crossings occur before all black or right ones. By the definition of the preorder, all these diagrams must have at least one left crossing, and thus the image lies in  $L$ .  $\square$

**Definition 1.13.** *We let  $S_{\mathbf{i}}^{\kappa} = P_{\mathbf{i}}^{\kappa}/L$  be the **standard module** for  $\kappa$  and  $\mathbf{i}$ .*

Below, when we speak of a **group** of black strands, we will always mean the set of black strands which originate between two consecutive red strands at the bottom of the diagram.

Consider the set  $\Phi$  of permutations of the bottom ends of the strands which only move black strands into blocks to their left and are minimal coset representatives for the permutations of the strands at the top of the diagram. We first give these a partial order which only depends only on the resulting idempotent at the top of the diagram.

So, we first preorder  $\Phi$  according to this preorder on the idempotent  $(\mathbf{i}_\phi, \kappa_\phi)$  which appears at the top of the diagram. Then within the permutations giving a single idempotent, we use the Bruhat order. Unlike the preorder above, this is a partial order.

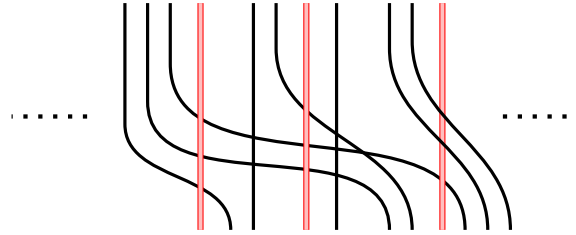


FIGURE 4. The element  $x_\phi$

Let  $x_\phi$  be an element where we permute the strands exactly according to a chosen reduced word of  $\phi \in \Phi$ . Let

$$P_{\leq \phi} = \langle x_{\phi'} | \phi' \leq \phi \rangle \subset P_{\mathbf{i}}^\kappa \quad P_{< \phi} = \langle x_{\phi'} | \phi' < \phi \rangle \subset P_{\mathbf{i}}^\kappa$$

While the element  $x_\phi$  is not unique, since it depends on a choice of reduced word, any two choices differ by an element of  $L_{< \phi}$ , so the filtration described above is unique.

**Proposition 1.14.**  $P_{\leq \phi} / P_{< \phi} \cong S_{\mathbf{i}_\phi}^{\kappa_\phi}$ .

We note that some of these subquotients are trivial, but in this case the corresponding standard module is trivial as well.

*Proof.* We have a map  $P_{\mathbf{i}_\phi}^{\kappa_\phi} \rightarrow P_{\leq \phi} / P_{< \phi}$  given by multiplication on  $x_\phi$ . This map is clearly surjective and factors through the standard quotient  $S_{\mathbf{i}_\phi}^{\kappa_\phi}$ . Thus, we need only show that it is injective.

Assume not. Then there is a diagram  $A$  such that

- $A$  represents a non-zero element of  $S_{\mathbf{i}_\phi}^{\kappa_\phi}$ , and
- $x_\phi A$  is contained in  $L_{< \phi}$

This containment must be realized by applying relations. Assume that at some point, we must pull a black strand left across a red one. We can “push down” this strand which goes further left than it does in  $x_\phi$  until it is past all other elements of  $A$  (correction terms will appear, but these will also have a black strand further left than in  $x_\phi$ , though it may connect to a different point at the bottom of the diagram).

Thus, this diagram begins with  $x_\phi$  times a left crossing, so it is zero in  $S_{i_\phi}^{\kappa_\phi}$  and thus we may simply drop this diagram from  $A$ , and it will retain its properties.

Thus, we must be able to realize this containment by never pulling a black strand left over a red one. In particular,  $x_\phi A$  must be in the submodule generated by  $x_{\phi'}$  where  $\phi'$  has the same endpoints at  $\phi$ , and is smaller in Bruhat order. But this is impossible, it would have to be accomplished using only black/black relations (any slide of a black strand over a red one would have to be counterbalanced by one in the opposite direction), and since  $\phi$  is a shortest coset representative on each side, there are no crossings between black strands in the same group at the top, since it is a shortest coset representative. Thus, black/black relations can only be applied to  $A$ , which contradicts the fact that it is non-zero in  $S_{i_\phi}^{\kappa_\phi}$ .  $\square$

**Corollary 1.15.** *The algebra  $E^\lambda$  is standardly stratified for the preorder on simples*

This preorder can be packaged as the dominance order for a function  $\alpha_{i,\kappa}: [1, \ell] \rightarrow X(\mathfrak{g})$  given by

$$\alpha_{i,\kappa}(k) = \sum_{\kappa(k-1) < j \leq \kappa(k)} \alpha_{i_j}$$

**Corollary 1.16.** *Every standard module has a finite length projective resolution.*

*Proof.* First note that if a module  $M$  is filtered by modules which have finite length projective resolutions, these resolutions can be glued to give a finite length resolution of the entire module.

Now, we induct on the partial order  $\leq$ . If a standard is maximal in this order, it is projective. For an arbitrary standard, there is a map  $P_\kappa^i \rightarrow S_\kappa^i$  with kernel filtered by standards higher in the partial order. Since each of these has a finite length projective resolution,  $S_\kappa^i$  does as well.  $\square$

We let  $\mathfrak{V}_\alpha^{\lambda_1, \dots, \lambda_n} = E_{\alpha(1)}^{\lambda_1} \otimes \dots \otimes E_{\alpha(\ell)}^{\lambda_\ell} - \text{mod}$ , and as before  $\mathcal{V}_\alpha^{\lambda_1, \dots, \lambda_n} = D^\beta(\mathfrak{V}_\alpha^{\lambda_1, \dots, \lambda_n})$ .

**Proposition 1.17.** *The subcategory  $\mathcal{C}^\alpha$  generated by  $S_i^\kappa$  for fixed  $\lambda, \alpha$  is equivalent to  $\mathfrak{V}_\alpha^{\lambda_1, \dots, \lambda_n}$ . The subcategories  $\mathcal{C}^\alpha$  form a semi-orthogonal decomposition of the category  $\mathfrak{V}_\alpha^\lambda$  with respect to dominance order.*

*Proof.* Of course, by Proposition 1.14, the subcategory generated by  $\mathcal{C}^{\alpha'}$  for  $\alpha' > \alpha$  in the dominance order is the same as that generated by  $P_i^\kappa$  such that  $\alpha_{i,\kappa} > \alpha$ . Since all the simple modules in  $S_i^\kappa$  are given by idempotents  $e_{i,\kappa}$  such that  $\alpha_{i,\kappa} \leq \alpha$ , we have

$$\text{Ext}^\bullet(S_{i'}^{\kappa'}, S_i^\kappa) = 0$$

whenever  $\alpha_{i,\kappa} < \alpha_{i',\kappa'}$ , and higher Ext's vanish when equality holds.

In particular, Let  $e_\alpha$  be the idempotent which is 1 on projectives  $P_i^\kappa$  with  $\alpha_{i,\kappa} = \alpha$ . We let  $S_\alpha$  be the standard quotient of the projective  $e_\alpha E^\lambda$ . Since  $S_\alpha$  generates  $\mathcal{C}^\alpha$ , and has trivial higher Exts

$$\mathcal{C}^\alpha \cong \text{End}^{op}(S_\alpha) - \text{mod.}$$

Let us calculate this endomorphism algebra. By the projective property, every endomorphism of  $S_\alpha$  is induced by an endomorphism of  $e_\alpha E^\lambda$ . Thus  $\text{End}^{op}(S_\alpha)$  is the quotient of the subalgebra of  $e_\alpha E^\lambda e_\alpha$  which preserves the kernel of the standard quotient modulo those that send everything to the kernel.

Proposition 1.2 implies that such an element can be written as a sum of diagrams where all left crossing come below all right ones. By the definition of the standard quotient such a diagram is 0 if it has any of the former type. On the other hand, an element of  $e_\alpha E^\lambda e_\alpha$  must have equal numbers of the two types of crossings, so our element can be "straightened" so that no red and black strands ever cross. Thus, we have a surjective map from  $R(\alpha(1)) \otimes \cdots \otimes R(\alpha(\ell))$  to this endomorphism ring.

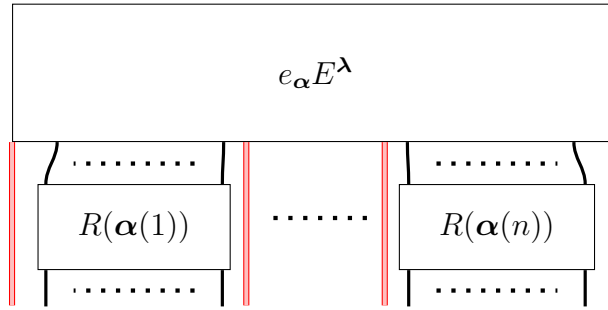


FIGURE 5. The action of  $R(\alpha(1)) \otimes \cdots \otimes R(\alpha(\ell))$  on  $e_\alpha E^\lambda$ .

Obviously, the cyclotomic ideal of this tensor product is killed by the map to  $\text{End}^{op}(S_\alpha)$ , so we have a surjective map  $E^{\lambda_1} \otimes \cdots \otimes E^{\lambda_\ell} \rightarrow \text{End}^{op}(S_\alpha)$ , which we need only show is also injective.

The kernel of this map are elements which send everything to the kernel for the standard quotient, which is the same as saying it sends each primitive idempotent not orthogonal to  $e_\alpha$  to the kernel of the map to the standard. Such an element can be expressed in terms of diagrams where all left crossings come before all right ones. Applying black/black relations will not change this property, so such a diagram can be only be obtained from one like that shown in Figure 5, and then dragging a red strand across a black one as shown in the first version of our relation (2) before applying any black/black relations. Thus, in the original form, our picture must contain a small piece that looks like the right hand side of that relation, which is *exactly* the minimal set of diagrams killed by imposing the cyclotomic relations. The

same argument shows that this map factors through the cyclotomic quotient, and thus

$$E_{\alpha(1)}^{\lambda_1} \otimes \cdots \otimes E_{\alpha(\ell)}^{\lambda_\ell} \cong \text{End}^{op}(S_\alpha). \quad \square$$

Thus, we can think of  $S_\alpha$  as a  $E_{\alpha(1)}^{\lambda_1} \otimes \cdots \otimes E_{\alpha(\ell)}^{\lambda_\ell} - E_\alpha^\lambda$ -bimodule, and  $S = \bigoplus_\alpha S_\alpha$  as a  $E^{\lambda_1} \otimes \cdots \otimes E^{\lambda_\ell} - E^\lambda$ -bimodule.

**Definition 1.18.** *The standardization functor is the right derived tensor product with this bimodule:*

$$\mathbb{S}^\lambda = - \overset{L}{\otimes}_{E^{\lambda_1} \otimes \cdots \otimes E^{\lambda_\ell}} S : \mathcal{V}^{\lambda_1; \dots; \lambda_\ell} \rightarrow \mathcal{V}^\lambda$$

More generally, we can construct partial standard modules, where we only kill the left crossings for some of the red strands. This will give us a standardization functor

$$\mathbb{S}^{\lambda_1; \dots; \lambda_m} : \mathcal{V}^{\lambda_1; \dots; \lambda_\ell} \rightarrow \mathcal{V}^\lambda$$

for any sequence of sequences  $\lambda_1, \dots, \lambda_m$  such that the concatenation  $\lambda_1 \cdots \lambda_m$  is equal to  $\lambda$ .

Of particular interest is the standardization functor which corresponds to adding a new red strand labeled  $\mu$  and no black ones, since this categorifies the inclusion of  $V_\lambda \otimes \{v_{high}\} \hookrightarrow V_\lambda \otimes V_\mu$ . We denote this functor  $\mathbb{S}^{\lambda; \mu}(- \boxtimes P_\emptyset) = \mathfrak{J}_\mu$ .

We can think of this standardization functor as a (very far from full) inclusion of the naive tensor product category into ours. This functor is full when only considered on objects landing in one piece of the semi-orthogonal decomposition, but there are, of course, many “new” maps between the different pieces of this decomposition.

**1.5. Self-dual projectives.** One interesting consequence of this Frobenius structure is the understanding it gives us of the self-dual projectives of our category. Self-dual projectives have played a very important role in understanding the structure of representation theoretic categories like  $\mathfrak{B}^\lambda$ ; for example, the unique self-dual projective in BGG category  $\mathcal{O}$  for  $\mathfrak{g}$  played a very important role in Soergel’s analysis of that category [Soe90, Soe92], and the self-dual projectives in category  $\mathcal{O}$  for a rational Cherednik algebra provide an important perspective on the Knizhnik-Zamolodchikov functor define by Ginzburg, Guay, Opdam and Rouquier [GGOR03]. In particular, as Mazorchuk and Stroppel show [MSb], these modules also play an important role in the identification of the Serre functor; we will apply their results in Section 2.3.

**Theorem 1.19.** *The following are equivalent:*

- (1) *The projective  $P_{\mathfrak{i}}^\kappa$  is self-dual.*
- (2) *The projective  $P_{\mathfrak{i}}^\kappa$  is injective.*
- (3) *Any simple quotient of  $P_{\mathfrak{i}}^\kappa$  appears in the socle of a projective.*
- (4) *We have  $\kappa = 0$ .*

Let  $P^0$  be the sum of all the projectives with  $\kappa = 0$ .

*Proof.* (1)  $\Rightarrow$  (2): The dual of a projective is always injective.

(2)  $\Rightarrow$  (3): If a projective is injective, its dual is projective-injective as well, and so its cosocle is the socle of a projective (by self-duality of simples).

(3)  $\Rightarrow$  (4): This is a simple calculation; adding right crossings at the top of a diagram never kills elements of a projective. It is equivalent to show that the maps induced between projectives by adding left crossings at the bottom are injective. By Proposition 1.14, such a map preserves standard filtrations, and in fact induces an isomorphism of each successive quotient of the target to those in the image (which do not include all successive quotients in the image).

(4)  $\Rightarrow$  (1): We must find a non-degenerate pairing  $P_{\mathbf{i}}^0 \otimes P_{\mathbf{i}}^0 \rightarrow \mathbb{k}$ . This is given by  $(a, b) = \text{tr}(ab)$ , where  $\text{tr}$  is the Frobenius trace on  $\text{End}(P^0) \cong E^\lambda$  given.  $\square$

For two rings  $A$  and  $B$ , we say a module  $M$  has the **double centralizer property** if  $\text{End}_B(M) = A$  and  $\text{End}_A(M) = B$ . In particular, this implies that the functor  $\otimes_A M : A\text{-mod} \rightarrow B\text{-mod}$  is fully faithful on projectives (it could be quite far from being a Morita equivalence, as the theorem below shows).

From (3) above and [MSb, Corollary 2.6], it follows immediately that

**Corollary 1.20.** *The projective-injective  $P^0$  has the double centralizer property for the actions of  $E^\lambda$  and  $E^\lambda$  on the left and right.*

Thus, our algebra can be realized as the endomorphisms of a collection of modules over the original KLR algebra, in a way analogous to the realization of category  $\mathcal{O}$  as the modules over endomorphisms of a particular module over the coinvariant algebra, or of the cyclotomic  $q$ -Schur algebra as the endomorphisms of a module over the Hecke algebra.

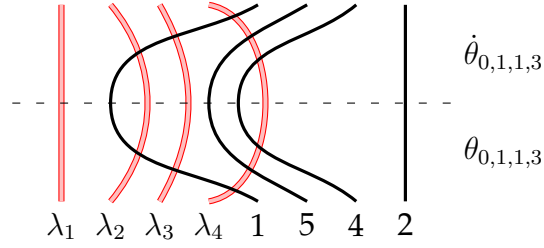
In fact, these modules are easy to identify. Given  $(\mathbf{i}, \kappa)$ , we consider the element  $y_{\mathbf{i}, \kappa}$  of  $P_{\mathbf{i}}^0$  given by

$$y_{\mathbf{i}, \kappa} = e_{\mathbf{i}} \prod_{j=1}^{\ell} \prod_{k=\kappa(j)+1}^n y_k^{\lambda_j^{i_k}}.$$

Pictorially this is given by multiplying the element with no black/black crossings going from  $(\mathbf{i}, 0)$  to  $(\mathbf{i}, \kappa)$  (which we denote  $\vartheta_{\kappa}$ ) by its horizontal reflection  $\dot{\vartheta}_{\kappa}$  and then straightening the strands.

**Proposition 1.21.** *The algebra  $E^\lambda$  is isomorphic to the algebra  $\text{End}_{E^\lambda}(\bigoplus_{\mathbf{i}, \kappa} y_{\mathbf{i}, \kappa} E^\lambda)$ .*

*Proof.* Based on Corollary 1.20, all we need to show is that  $\text{Hom}_{E^\lambda}(P^0, P_{\mathbf{i}}^\kappa) \cong y_{\mathbf{i}, \kappa} P_{\mathbf{i}}^0$  as a  $E^\lambda$  representation. A map  $m$  from  $P_{\mathbf{i}'}^0$  to  $P_{\mathbf{i}}^\kappa$  is simply a linear combination of diagrams starting at  $\mathbf{i}'$  with the correct placement of red strands and ending at  $\mathbf{i}$  with all red strands to the right. With such a diagram, we can assure that all red/black crossings occur above all black/black ones, so  $m = \vartheta_{\kappa} m'$ , where  $m' \in E^\lambda$ .


 FIGURE 6. The element  $y_{(1,5,4,2),(0,1,1,3)}$ .

Thus, we have maps

$$\mathrm{Hom}_{E^\lambda}(P^0, P_{\mathbf{i}}^0) \xrightarrow{\dot{\vartheta}_\kappa} \mathrm{Hom}_{E^\lambda}(P^0, P_{\mathbf{i}}^\kappa) \xrightarrow{\dot{\vartheta}_\kappa} \mathrm{Hom}_{E^\lambda}(P^0, P_{\mathbf{i}}^0)$$

given by composition; the first of these is surjective, as we argued above, and the latter injective, since pulling red strands toward the left never kills an element, since the map induces an isomorphism on all successive quotients of the filtration by standards which lie in the image as we argued in the proof of Theorem 1.19. Thus,  $\mathrm{Hom}_{E^\lambda}(P^0, P_{\mathbf{i}}^\kappa)$  is isomorphic to the image of the composition of these maps, which is  $y_{\mathbf{i}, \kappa} E^\lambda$ .  $\square$

**1.6. Decategorification.** The sum of all the standardization functors induces a map of Grothendieck groups

$$K^0(\mathcal{V}^{\lambda_1}) \otimes \cdots \otimes K^0(\mathcal{V}^{\lambda_\ell}) \cong V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_\ell} \rightarrow K^0(\mathcal{V}^\lambda).$$

**Theorem 1.22.** *This map is an isomorphism of  $U_q(\mathfrak{g})$ -representations*

$$K^0(\mathcal{V}^\lambda) \cong V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_\ell}.$$

where  $K^0(\mathcal{V}^\lambda)$  inherits a  $U_q(\mathfrak{g})$  action from  $\mathfrak{F}_i$  and  $\mathfrak{E}_i$ .

*Proof.* This is an isomorphism of vector spaces since it is known to be on each piece of the semi-orthogonal decomposition discussed above, by Theorem 1.5.

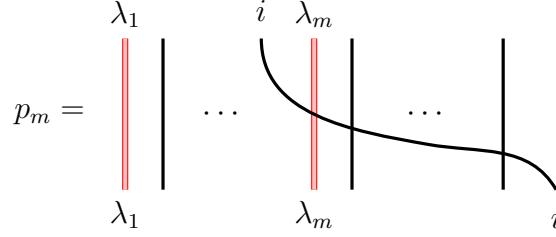
Consider a Hecke operator applied to a standard module  $\mathfrak{F}_i S_{\mathbf{i}}^\kappa$ . This module is very close to being a standard module, but we have not quotiented by the operation of moving the rightmost strand past red strands. Thus, we consider the filtration consisting of submodules generated by

We let  $\kappa_m$  and  $\mathbf{i}_m$  be associated to the sequence pictured at the top of Figure 7. Of course  $\langle p_m \rangle \subset \langle p_{m+1} \rangle$ , and we have an isomorphism of the successive quotients

$$\langle p_{m+1} \rangle / \langle p_m \rangle \cong S_{\mathbf{i}_m}^{\kappa_m} \left( - \sum_{j=m+1}^{\ell} \langle \alpha_i, \lambda_j - \alpha(j) \rangle \right)$$

This is a direct categorification of the formula for the coproduct

$$\Delta(F_i) = F_i \otimes \tilde{K}_{-i} + 1 \otimes F_i,$$


 FIGURE 7. The element  $p_m$ 

and thus establishes that the induced action is the tensor product.

On the other hand, consider  $\mathfrak{E}_i S_1^\kappa$ . Consider the filtration on this where  $q_i$  is the submodule generated by the collection of diagrams where the rightmost strand at the top lands to the right of the  $i$ th strand and  $i + 1$ st at the bottom. Then we have an isomorphism

$$(1.4) \quad q_i/q_{i+1} \cong S_{\mathfrak{im}}^{\kappa_m} \left( \sum_{j=1}^m \langle \alpha_i, \lambda_j - \alpha(j) \rangle - 1 \right) \\ \cong \mathbb{S}^\lambda(\cdots \boxtimes P_{\mathfrak{im}-1} \boxtimes \mathfrak{E}_i P_{\mathfrak{im}} \boxtimes P_{\mathfrak{im}+1} \boxtimes \cdots) \left( \sum_{j=1}^{m-1} \langle \alpha_i, \lambda_j - \alpha(j) \rangle \right)$$

so this filtration categorifies the identity

$$\Delta(E_i) = E_i \otimes 1 + \tilde{K}_i \otimes E_i. \quad \square$$

*Remainder of Proof of 1.6.* Since  $\mathfrak{F}_i$  generates the category  $\mathcal{V}^\lambda$ , we need only check that this functor preserves finite total cohomology. It is enough to prove that as if  $E^\lambda$  is considered as a  $E^\lambda$  module using  $\nu_i$ , it has a finite length projective resolution. This follows since, as we argue in the proof of Theorem 1.22, this module has a standard filtration, and standards have finite length resolutions.  $\square$

## 2. BRAIDING AND RIGIDITY FUNCTORS

We now proceed to construct the functors whose existence is claimed in Theorem B. These correspond to the structures on a braided rigid ribbon category. Alternatively, these have a topological interpretation; we imagine taking the red strands of our category, placing them in  $\mathbb{R}^3$  and thickening them to ribbons (so that we keep track of twists of them). Then our functors correspond to the following operations on ribbons:

- Adding a full twist to one of the ribbons: the corresponding operator in the quantum group is called the **ribbon element**.
- Crossing two ribbons: the corresponding operator in representations of the quantum group is called the **braiding** or **R-matrix**.

- Creating a cup, or closing a cap: the corresponding operators in representations of the quantum group are called the **coevaluation** and **quantum trace**.

2.1. **Ribbon structure.** The easiest of these is the **ribbon functor**, though it has a slight subtlety; we are forced by the more rigid structure to pick a different ribbon element in  $U_q(\mathfrak{g})$  than is conventionally used. Thus, we choose the ribbon element in  $U_q(\mathfrak{g})$  uniquely determined by the fact that it acts on the simple representation of highest weight  $\lambda$  by  $(-1)^{\rho^\vee(\lambda)} q^{\langle \lambda, \lambda \rangle + 2\langle \lambda, \rho \rangle}$ . This element is constructed by Snyder and Tingley in [ST]; see Theorem 4.6 of that paper for a proof that this is a ribbon element.

The exponent is not always an integer, but of course, there is no harm in allowing our grading to be rational. Our ribbon functor is a straightforward categorification of this.

**Definition 2.1.** *The ribbon functor associated to a strand labeled with  $\lambda$  is the grading shift*

$$X \mapsto X(\langle \lambda, \lambda \rangle + 2\langle \lambda, \rho \rangle)[2\rho^\vee(\lambda)].$$

Due to the extra trouble of drawing ribbons, we will draw all pictures in the blackboard framing.

This different choice of ribbon element will not seriously affect invariance, but we will leave consideration of this point until Section 3.

2.2. **Braiding.** Next, we describe the braiding in terms of an explicit bimodule  $\mathfrak{B}_\sigma$  attached to each braid. Let us first describe the bimodule  $\mathfrak{B}_{\sigma_k}$  attached to a single positive crossing of the  $k$ th and  $k + 1$ st strands.

Like the algebra  $E^\lambda$ , the bimodule  $\mathfrak{B}_{\sigma_k}$  is spanned by pictures. In fact, it is spanned by pictures which are identical to those used in the definition of  $E^\lambda$ , except that we must have a single crossing between the  $k$ th and  $k + 1$ st red strands. These pictures are acted upon on the left by  $E^\lambda$  and on the right by  $E^{\sigma_k \lambda}$  in the obvious way. This obtains a grading in as usual, but with the red crossing given degree  $-\langle \lambda_k, \lambda_{k+1} \rangle$

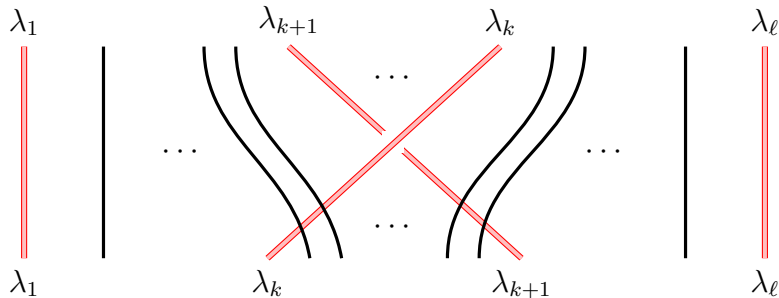
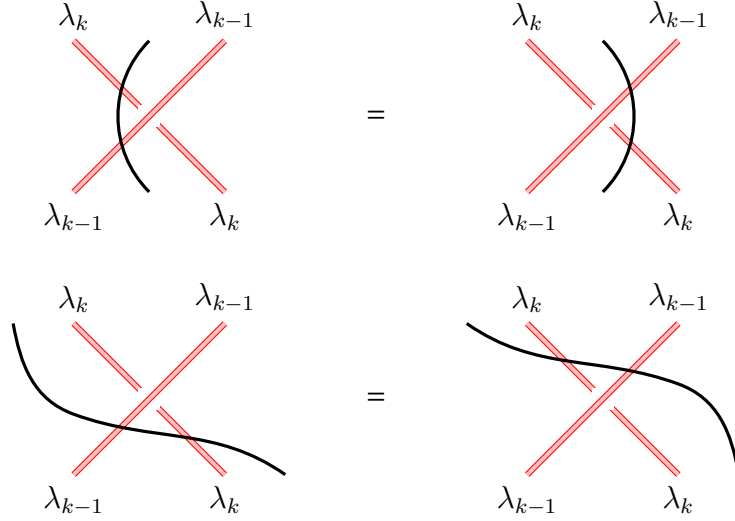


FIGURE 8. An example of an element of  $\mathfrak{B}_{\sigma_k}$ .

As before, we need to mod out by relations:

- We impose all local relations that we had before, including planar isotopy.
- Furthermore, we have to add the relations



**Definition 2.2.** Let  $\mathbb{B}_{\sigma_k}$  be the functor  $- \otimes^L \mathfrak{B}_{\sigma_k}$ .

First, we require some technical results about this functor. Recall that for any permutation  $w$ , there is a unique positive braid  $\sigma_w$  which induces that permutation on the ends of the strands of the same length of the permutation, constructed by a picking a reduced expression  $w = s_{i_1} \cdots s_{i_m}$ , and taking the product  $\sigma_w = \sigma_{i_1} \cdots \sigma_{i_m}$ . We call this the permutation's **minimal lift**.

**Lemma 2.3.** If  $\sigma = \sigma_{i_1} \cdots \sigma_{i_m}$  is a minimal lift of a permutation, then the functor  $\mathbb{B}_\sigma = \mathbb{B}_{\sigma_{i_1}} \cdots \mathbb{B}_{\sigma_{i_m}}$  is independent of the choice of reduced word (up to canonical isomorphism).

For any projective  $\mathbb{B}_\sigma(P_i^\kappa)$  has a standard filtration and  $\mathbb{B}_\sigma(S_i^\kappa)$  is a module (that is,  $\text{Tor}_{E^\lambda}^{>0}(S_i^\kappa, \mathfrak{B}_\sigma) = 0$ ).

In particular,  $\mathbb{B}_\sigma$  deserves the finite total cohomology derived category  $D^\beta$ .

*Proof.* First, we note that assuming the first two paragraphs of the theorem, we find that  $\mathfrak{B}_\sigma$  considered as a left module (which is the same as  $\mathfrak{B}_\sigma$ ) has a finite length free resolution. So any module  $M$  is sent to a finite length complex  $M \otimes^L \mathfrak{B}_\sigma$ , which is thus in  $D^\beta$ .

We prove this by induction on the length of  $w$ . This induction is slightly subtle, so rather than attempt each step in one go, we break the theorem into 3 statements, and induct around a triangle. Consider the three statements (for each positive integer  $n$ ):

- $p_n$  : For all  $\sigma$  with  $\ell(\sigma) = n$ ,  $\mathbb{B}_\sigma$  sends projectives to modules.
- $f_n$  : For all  $\sigma$  with  $\ell(\sigma) = n$ ,  $\mathbb{B}_\sigma$  sends projectives to objects with standard filtrations, and is independent of reduced expression.
- $s_n$  : For all  $\sigma$  with  $\ell(\sigma) = n$ ,  $\mathbb{B}_\sigma$  sends standards to modules.

Our induction proceeds by showing

$$\cdots \Rightarrow p_n \Rightarrow f_n \Rightarrow s_n \Rightarrow p_{n+1} \Rightarrow \cdots$$

These are all obviously true for  $\sigma = 1$ , so this covers the base of our induction.

$f_n \Rightarrow s_n$ : Note that  $\text{Tor}^i(S_i^\kappa, \dot{S}_i^{\kappa'}) = 0$  for all  $i > 0$ , since there is always a subcategory of representations containing both in which one of them is projective, given by taking the span of all pieces of the semi-orthogonal decomposition smaller than either one of them.

Let  $\bar{\sigma}$  be a reduced positive braid for the inverse of  $\sigma$ . Then if we let  $\mathfrak{B}_{\bar{\sigma}}$  be  $\mathfrak{B}_\sigma$  with the left and right actions reversed by the dot-anti-automorphism, then  $\mathfrak{B}_{\bar{\sigma}} \cong \mathfrak{B}_\sigma$ .

By  $f_n$ , the bimodule  $\mathfrak{B}_{\bar{\sigma}}$  has a standard filtration as a right module, so  $\mathfrak{B}_\sigma$  has a standard filtration as a left module. Thus, we have  $\text{Tor}^i(S_i^\kappa, \mathfrak{B}_\lambda^{\lambda'})$  for  $i > 0$  and the same holds for any module with a standard filtration.

$s_n \Rightarrow p_{n+1}$ : We can write  $\mathbb{B}_\sigma = \mathbb{B}_{\sigma'} \mathbb{B}_{\sigma''}$  where  $\sigma', \sigma''$  are of length  $< n + 1$ . Thus, by assumption,  $\mathbb{B}_{\sigma''}$  sends projectives to standard filtered modules, and  $\mathbb{B}_{\sigma'}$  sends standards to modules. The result follows.

$p_n \Rightarrow f_n$ : Since  $\mathbb{B}_\sigma$  sends projectives to modules, the bimodule  $\mathfrak{B}_\sigma$  is the naive tensor product of those corresponding to individual crossings. The commutation of crossings with no common strands is clear. In order to do a Reidemeister III move, note that any bunch of 3 red strands which does a full twist can have its “triangle” entirely cleared if black strands (since any black strand passing through the triangle must touch two of the sides, and thus can be slid through the place where they cross). The isomorphism is given by simply doing Reidemeister III on the red strands, which interferes with no black ones.

Now, we construct the standard filtration on  $D = \mathbb{B}_\sigma P_1^\kappa$ . Consider the set  $\sigma(\Phi)$  where  $\Phi$  is the parameter set of the standard filtration on the projective. We act on these permutations by letting each group of black strands move with the red strand to its left. As before, we can place a partial order on these by consider the preorder on the labeling of the tops of the strands, and then within each labeling using the Bruhat order. The element  $y_\phi$  which we attach to this element is again the diagram which permutes the red and black strands according to a reduced word of the permutation.

As before, we attach to this a filtration  $D_{\leq \phi}, D_{< \phi}$  to these elements and partial order. Multiplication by  $y_\phi$  gives a surjection  $S_{i_\phi}^{\kappa_\phi} \twoheadrightarrow D_{\leq \phi} / D_{< \phi}$ , which we aim to show is an isomorphism.

The proof is essentially the same as Proposition 1.14: if at any point one pulls a black strand left over a red one, the corresponding diagram is 0 in the standard quotient, so one can only use black/black relations, which show that the diagram one multiplied by must have been 0 in the standard quotient.  $\square$

Let  $\tau$  be a positive lift of the longest element. This is essentially a half twist, but with the blackboard framing, not the one with ribbon half-twists as well.

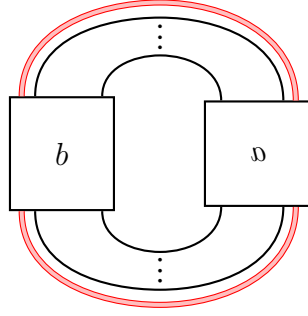


FIGURE 9. The pairing

Recall that a module  $M$  over a standardly stratified algebra is called **tilting** if  $M$  and  $M^*$  both have standard filtrations<sup>2</sup>.

**Corollary 2.4.** *The modules  $\mathbb{B}_\tau P_i^\kappa$  are tilting, and every indecomposable tilting module is a summand of these tiltings.*

*Proof.* We show first that  $\mathbb{B}_\tau P_i^\kappa$  is self-dual. The pairing that achieves this duality is a simple variant on that described in Section 1.3, where as before, we form a closed diagram and evaluate its constant term. This pairing is pictorially represented in Figure 9.

The non-degeneracy of this pairing follows from that on  $P_i^0$ . We have already noted that  $P_i^\kappa$  has an embedding into  $P_i^0$  consistent with the standard filtration, given by left multiplication by the element  $\theta_\kappa$ . By the Tor-vanishing of  $\mathfrak{B}_\tau$  and any module with a standard filtration, this map induces an inclusion  $\mathbb{B}_\tau P_i^\kappa \rightarrow P_i^0$ .

We also have a map  $P_i^0 \rightarrow P_i^\kappa$  given by  $\theta_{\kappa, i}$ ; while this map is not surjective, the induced map  $P_i^0 \rightarrow \mathbb{B}_\tau P_i^\kappa$  is, since any non-zero diagram in  $\mathbb{B}_\tau P_i^\kappa$  can be drawn with a section in the middle where all black strands are right of all red strands (apply Proposition 1.2, with all permutations given the reduced word that pulls red strands to the left before reordering any black ones).

The pairing of Figure 8 is that induced by these maps, which in particular shows that the perpendicular to the image of the inclusion contains the kernels of the surjection. Since these have the same dimension, they coincide and the pairing is non-degenerate. Thus,  $\mathbb{B}_\tau P_i^\kappa$  is self-dual.

By Lemma 2.3,  $\mathbb{B}_\tau P_i^\kappa$  has a filtration by standards, so by self-duality, it also has a filtration by costandards and thus is tilting.

For each indecomposable summand of a costandard, there is at most one indecomposable tilting with a surjection to this costandard, and all tiltings are of this form.

<sup>2</sup>This is stronger than the notion of tilting defined in [MSb]. In the terminology of that paper, this notion would be “tilting and cotilting.” In this case, we are lucky and there is no difference between tilting and cotilting modules.

Since all costandards appear as quotients of  $\mathbb{B}_\tau P_i^\kappa$ , since shows that all indecomposable tiltings occurs as summands in these.  $\square$

**Proposition 2.5.**  $\mathbb{B}_j(\mathbb{S}^\lambda(P_{\dots; i_j; \emptyset; \dots})) \cong \mathbb{S}^\lambda(P_{\dots; \emptyset; i_j; \dots}) \left( \langle \lambda_j - \alpha(j), \lambda_{j+1} \rangle \right)$

*Proof.* This follows immediately from the fact that  $\mathbb{B}_j(\mathbb{S}^\lambda(P_{\dots; i_j; \emptyset; \dots}))$  is generated by the single diagram shown in Figure 10.  $\square$

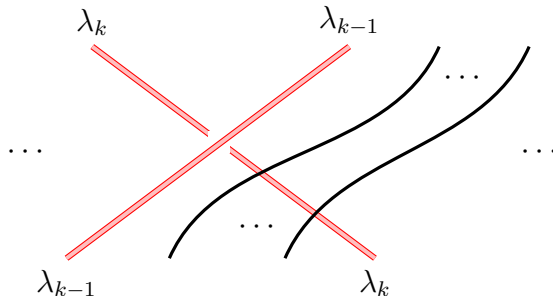


FIGURE 10. The generator of  $\mathbb{B}_j(\mathbb{S}^\lambda(P_{\dots; i_j; \emptyset; \dots}))$ .

**Corollary 2.6.** *The action of  $\mathbb{B}_\sigma$  categorifies the action of the braiding.*

*Proof.* First, note that  $\mathbb{B}_\sigma$  commutes with the Hecke functors, so the induced action on  $V_\lambda$ , which we denote by  $R_\sigma$ , commutes with the action of  $U_q(\mathfrak{g})$ . Thus we need only calculate the action of  $R_\sigma$  on a pure tensor of a weight vectors with a *highest* weight vector  $v_h$  in the  $j + 1$ st place.

The space of such vectors is spanned by the classes of the form  $\mathbb{S}^\lambda(P_{\dots; i_j; \emptyset; \dots})$ . Thus, Proposition 2.5 implies that

$$R_\sigma(v_1 \otimes \dots \otimes v_j \otimes v_h \otimes \dots \otimes v_\ell) = q^{\langle \text{wt}(v_j), \lambda_{j+1} \rangle} v_1 \otimes \dots \otimes v_h \otimes v_j \otimes \dots \otimes v_\ell$$

which is exactly what the braiding does to vectors of this form by [CP95, Theorem 8.3.9]. Since vectors of this form generate the representation, there is a unique endomorphism with this behavior, and  $R_\sigma$  is the braiding.  $\square$

**Theorem 2.7.** *The functor  $\mathbb{B}_\sigma$  is an equivalence which only depends on the element in the braid group, and thus gives a weak braid groupoid action on our categories.*

*Proof.* First we check that the braid relations are satisfied up to isomorphism of functors. But the braid relations only involve positive braids that are minimal lifts of permutations, which we have already proven are independent of their expression in braid generators.

Thus the only difficulty is in showing that  $\mathbb{B}_\sigma$  is a derived equivalence. We will first show this for  $\mathbb{B}_\tau$ .

The higher Ext's between tilting modules always vanish so we always have that  $\text{Ext}^{>0}(\mathbb{B}_\tau P_{\mathbf{i}}^\kappa, \mathbb{B}_\tau P_{\mathbf{i}'}^{\kappa'}) = 0$ .

Let  $\varepsilon([A], [B]) = \sum_i (-1)^i \dim \text{Ext}^i(A, B)$  denote the Euler form on any of the Grothendieck groups we have considered. It follows from Corollary 2.6 that

$$\dim \text{Hom}(\mathbb{B}_\tau P_{\mathbf{i}}^\kappa, \mathbb{B}_\tau P_{\mathbf{i}'}^{\kappa'}) = ([\mathbb{B}_\tau P_{\mathbf{i}}^\kappa], [\mathbb{B}_\tau P_{\mathbf{i}'}^{\kappa'}]) = ([P_{\mathbf{i}}^\kappa], [P_{\mathbf{i}'}^{\kappa'}]) = \dim \text{Hom}(P_{\mathbf{i}}^\kappa, P_{\mathbf{i}'}^{\kappa'}).$$

The functor  $\mathbb{B}_\tau$  induces a map

$$\text{Hom}(P_{\mathbf{i}}^\kappa, P_{\mathbf{i}'}^{\kappa'}) \longrightarrow \text{Hom}(\mathbb{B}_\tau P_{\mathbf{i}}^\kappa, \mathbb{B}_\tau P_{\mathbf{i}'}^{\kappa'}).$$

This is injective, since no element of it kills the element which pulls all black strands to the right of all red strands below all crossings. Thus, it is surjective by the dimension calculation above. It follows that  $\mathbb{B}_\tau$  is an equivalence. Since it factors through any  $\mathbb{B}_{\sigma_k}$  on the left and right,  $\mathbb{B}_{\sigma_k}$  is an equivalence as well.  $\square$

Recall that the **Ringel dual** of a standardly stratified category is the modules over the endomorphism ring of a tilting generator, that is, the dual category to the heart of the  $t$ -structure in which the tiltings are projective. Let  $\bar{\lambda} = (\lambda_\ell, \dots, \lambda_1)$  be the reversal of the sequence  $\lambda$ .

**Corollary 2.8.** *The Ringel dual of  $\mathfrak{A}^\lambda$  is equivalent to  $\mathfrak{A}^{\bar{\lambda}}$ .*

If  $C_i$  and  $C'_i$  are semi-orthogonal decompositions indexed by  $i \in [1, n]$  then  $C'_i$  is the **mutation** of  $C_i$  by a permutation  $\sigma$  if the category generated by  $C_i$  for  $i \leq j$  is the same as that generated by  $C'_{\sigma(i)}$  for  $i \leq j$ .

**Proposition 2.9.** *For any braid  $\sigma$ ,  $\mathbb{B}_\sigma$  sends the canonical semi-orthogonal decomposition to its mutation by  $\sigma$ .*

*Proof.* It's clear from the standard filtration on projectives that the correct Ext vanishing holds and that

$$\mathbb{B}_\sigma S_{\mathbf{i}}^\kappa \equiv \mathbb{B}_\sigma P_{\mathbf{i}}^\kappa \equiv S_{\mathbf{i}'}^{\kappa'} \text{ modulo smaller } S_{\mathbf{i}}^\eta$$

where  $\kappa'$  and  $\mathbf{i}'$  are arrived at by moving the  $i$ th red strand and all black strands between that and the  $(i+1)$ -st rightward to the immediate left of the  $(i+2)$ -nd.  $\square$

**2.3. Serre functors.** It is a well-supported principle (see, for example, Mazorchuk and Stroppel [MSb]) that for any suitable braid group action on a category, the Serre functor will be given by the full twist. Here the same is true, up to grading shift. Let  $\mathfrak{R}$  be the functor given by a full positive twist of the red strands as **ribbons**, so compared to  $\mathbb{B}_\tau^2$ , we have also added a full twist of each individual ribbon. Let  $\mathfrak{S}'$  be the functor sends  $M \in \mathcal{V}_\alpha^\lambda$  to  $M(-2\langle \lambda, \rho \rangle - \langle \alpha, \alpha \rangle)[-2\rho^\vee(\lambda)]$ .

**Proposition 2.10.** *The right Serre functor of  $\mathcal{V}^\lambda$  is given by  $\mathfrak{S} = \mathfrak{R}\mathfrak{S}'$ .*

*Proof.* First consider the action of  $\mathfrak{S}$  on projective-injectives: the twists of red strands are irrelevant to black strands that begin to the right of all of them, so

$$\mathfrak{R} \cong \text{Id} \left( \langle \lambda, \lambda \rangle + 2\langle \lambda, \rho \rangle \right) [2\rho^\vee(\lambda)]$$

as functors on the projective-injective category. Since  $I_1^0 \cong P_1^0(\langle \lambda, \lambda \rangle - \langle \alpha, \alpha \rangle)$ , on this subcategory  $\mathfrak{S}P_1^0 = P_1^0(\langle \lambda, \lambda \rangle - \langle \alpha, \alpha \rangle) \cong I_1^0$  and so  $\mathfrak{S}$  is the graded Serre functor.

Since they both have costandard and standard filtrations in the same class in the Grothendieck group, we have that  $\mathbb{B}_\tau^{-1}I_1^\kappa$  and  $\mathbb{B}_\tau P_1^\kappa$  are the same self-dual tilting module (ignoring grading for the moment). Thus,  $\mathfrak{R}P_1^\kappa \cong I_1^\kappa$  (again, ignoring the grading). In particular,  $\mathfrak{R}$  sends projectives to injectives, and is an equivalence. By [MSb, Theorem 3.4], the result follows.  $\square$

**2.4. Coevaluation and quantum trace for a pair of representations.** Now, we must consider the cups and caps in our theory. The most basic case of this is  $\lambda = (\lambda, \lambda^*)$ , where we use  $\lambda^* = -w_0\lambda$  to denote the highest weight of the dual representation to  $V_\lambda$ .

We will be interested in the coevaluation, which is the canonical inclusion  $\mathbb{C} \rightarrow V_\lambda \otimes V_\lambda^*$  where the duality between  $V_\lambda$  and  $V_\lambda^*$  pairs the unique canonical basis vectors in the highest and lowest weight spaces to 1 (we must be careful about the isomorphism between  $V_\lambda^*$  and  $V_\lambda^*$ ). This map sends  $1 \in \mathbb{C}$  to the canonical basis vector in the invariants of  $V_\lambda \otimes V_\lambda^*$ .

Now, let us construct the representation which corresponds to this vector: pick a reduced word for the longest element of the Weyl group of  $\mathfrak{g}$ , and let  $s_j$  be the product of the first  $j$  reflections in this word. Consider the sequence

$$\mathbf{i}^\lambda = (i_1^{(\lambda^{i_1})}, i_2^{((s_1\lambda)^{i_2})}, \dots, i_k^{((s_{k-1}\lambda)^{i_k})})$$

**Proposition 2.11.** *The projective  $P_{\mathbf{i}^\lambda}^0$  over  $E^\lambda$  is irreducible.*

*Proof.* We will prove a slight stronger result: that for any reduced expression  $s$  in simple reflections, the corresponding projective is irreducible. The base case is just showing this when the expression is length 1, which is the case of  $\mathfrak{sl}_2$ , which was shown by Lauda [Lau] (this corresponds to the fact that the Grassmannian of  $k$ -planes in  $k$ -space is a point).

In general, we must show that  $\text{Hom}(P_{\mathbf{i}^\lambda}^0, P_{\mathbf{i}^\lambda}^0) = 1$ , since there is only one simple and one irreducible projective in this category. Thus, we need only consider diagrams beginning and ending with our preferred idempotent. We claim that such diagrams can be written as a sum of diagrams where no lines of different colors cross. This obviously reduces our proposition to the  $\mathfrak{sl}_2$  case.

Now consider an arbitrary diagram, and consider the left-most block of strands of a single color whose members cross strands of other colors. If no strands start in this block at the bottom and end up in a different block at the top, then we can simply “pull straight” and have a diagram where the first “bad block” is further right.

If a strand does leave this block traveling upward, it must be matched by one which leaves it traveling downward, and the strands must cross. Using RIII moves, one can move this crossing left (with correction terms that have fewer such strands, since the correction terms smooth crossings), so that all differently colored strands pass to its left. But then at this crossing, we have reordered the strands so that we get  $i_\lambda^{s'}$  for some truncation of our word, and then a repetition of the last element. This is a composition of Hecke functors corresponding to an empty weight space, so is 0. Thus, by induction, we are done.  $\square$

We can now consider this representation standardized in two different ways, obtaining two standard modules:  $S_{i_\lambda}^{(0, 2\rho^\vee(\lambda))} = P_{i_\lambda}^{(0, 2\rho^\vee(\lambda))}$  and  $S_{i_\lambda}^0$ . Proposition 2.11 shows that the first has simple cosocle and the second is itself simple. We denote the cosocles of these representations by  $L_\lambda$  and  $M_\lambda$ .

Let  $\mathbb{K}_\emptyset^{\lambda, \lambda^*} : \text{Vect} \rightarrow \mathcal{V}^{\lambda, \lambda^*}$  be the functor sending the vector space  $\mathbb{C}$  to  $L_\lambda$ .

Recall that coevaluation is the map  $K : \mathbb{C} \rightarrow V_\lambda \otimes V_\lambda^*$  sending 1 to the canonical element of the tensor product. By choosing an isomorphism  $V_\lambda^* \cong V_{\lambda^*}$ , this can be sent to any element invariant under the action of  $U_q(\mathfrak{g})$ .

**Proposition 2.12.** *The functor  $\mathbb{K}_\emptyset^{\lambda, \lambda^*}$  categorifies the coevaluation.*

*Proof.* Of course, the invariants are in the 0 weight space, and in fact, is the subspace  $\{v | E_i v = 0\}$  for any  $i$ . Of course,  $\mathfrak{E}_i L_\lambda = 0$  for all  $i$ , so the class of  $L_\lambda$  is in this subspace. In fact,  $L_\lambda$  is the only such representation, since the  $-\lambda^*$ -weight space of  $V_\lambda$  is 1 dimensional.  $\square$

We will also be interested in the “quantum trace”  $V_\lambda \otimes V_{\lambda^*} \rightarrow \mathbb{C}$ . This is not given by the dual of the coevaluation on  $V_{\lambda^*} \otimes V_\lambda$ , but is related to it by the ribbon structure.

**Proposition 2.13.** *The quantum trace is categorified by the functor*

$$\mathbb{T}_\emptyset^{\lambda, \lambda^*} = \text{Ext}^\bullet(L_\lambda, -)(2\langle \lambda, \rho \rangle)[2\rho^\vee(\lambda)] : \mathcal{V}^{\lambda, \lambda^*} \rightarrow \text{Vect}.$$

*Proof.* Here we use Theorem 3.2. While this theorem appears in a later section, it depends in no way on calculations of decategorification, so there is no circularity. This theorem shows that the functor  $\mathbb{T}_\emptyset^{\lambda, \lambda^*} \circ \mathbb{B}_{1,2} \circ \mathbb{R}_1$  satisfies the “S-move” with  $\mathbb{K}_\emptyset^{\lambda, \lambda^*}$ , and thus categorifies the evaluation. By the usual relations of ribbon categories (see Figure 11), the result follows.  $\square$

This, on its own, implies a homological realization of the quantum dimension of a representation as the graded dimension of an algebra.

**Corollary 2.14.** *The invariant of a circle is  $\text{Ext}^\bullet(L_\lambda, L_\lambda)(2\langle \lambda, \rho \rangle)[2\rho^\vee(\lambda)]$ , and the graded Euler characteristic of this space is the quantum dimension of  $V_\lambda$ .*

We recall that we have chosen a non-standard ribbon element, so that this quantum dimension differs from the usual one by a factor of  $(-1)^{2\rho^\vee(\lambda)}$ . If we did not

choose this slightly unusual convention, we would not be able to attach a self-dual bigraded vector space to the unknot.

In order to analyze the structure of  $L_\lambda$  and  $M_\lambda$ , we must understand some projective resolutions of standards. Let  $\kappa_i(1) = i, \kappa_i(0) = 0$ . Given a subset  $S \subset [i + 1, n]$ , we let  $\mathbf{i}_S$  be the sequence given by  $i_1, \dots, i_i$  followed by  $S$  in reversed sequence, and then  $[i + 1, n] \setminus S$  in sequence and let  $\kappa_S(1) = i + \#S$ . Let

$$\chi_S = \sum_{j \in S} \left\langle \alpha_{i_j}, \lambda^* - \sum_{\ell \leq j} \alpha_{i_\ell} \right\rangle.$$

**Proposition 2.15.** *The standard  $S_{\mathbf{i}_S}^{\kappa_j}$  has a projective resolution given by  $S_{\mathbf{i}_S}^{\kappa_S}(\chi_S)$  in degree  $|S|$ .*

*Proof.* We induct on  $n - i$ . If  $j = n$ , then  $S_{\mathbf{i}}^{\kappa_i}$  is itself projective, so we may take the trivial resolution. Let  $\mathbf{i}'$  be  $\mathbf{i}$  with its last entry removed, and  $\mathbf{i}''$  be  $\mathbf{i}$  with its last entry moved to the  $j + 1$ st position. As we noted in the proof of Theorem 1.22, we have an exact sequence

$$0 \longrightarrow S_{\mathbf{i}_{\{n\}}}^{\kappa_{\{n\}}}(\chi_{\{n\}}) \longrightarrow \mathfrak{F}_{i_n} S_{\mathbf{i}'}^{\kappa_j} \longrightarrow S_{\mathbf{i}}^{\kappa_j} \longrightarrow 0.$$

Applying the inductive hypothesis, we obtain projective resolutions of the left two factors. Furthermore, we can lift the leftmost map to a map between projective resolutions. The cone of this map is the desired projective resolution of  $S_{\mathbf{i}}^{\kappa_j}$ .  $\square$

This provides a resolution of  $M_\lambda$ , since it is itself standard. In particular, it shows that

**Corollary 2.16.**  $\text{Ext}^i(M_\lambda, L_\lambda) = \begin{cases} 0 & i \neq \langle \lambda, \rho \rangle \\ \mathbb{k}(2\langle \lambda, \rho \rangle) & i = \langle \lambda, \rho \rangle \end{cases}.$

*Proof.* All of the projectives which appear in the resolution of  $M_\lambda$  has no maps to  $L_\lambda$  except the last term which is  $P_{\mathbf{i}_\lambda}^{(0, \rho^\vee(\lambda))}$ . Thus we have

$$\text{Ext}^i(M_\lambda, L_\lambda) \cong \text{Ext}^{i-2\rho^\vee(\lambda)}(P_{\mathbf{i}_\lambda}^{(0, 2\rho^\vee(\lambda))}(-2\langle \lambda, \rho \rangle), L_\lambda)$$

and the result follows.  $\square$

This calculation is also important for showing how  $L_\lambda$  behaves under braiding.

**Proposition 2.17.**  $\mathbb{B}_{\sigma_1} L_\lambda \cong L_{\lambda^*}[-2\rho^\vee(\lambda)](-2\langle \lambda, \rho \rangle - \langle \lambda, \lambda \rangle).$

*Proof.* Unless  $\mathbf{i}$  is a sequence corresponding to weight 0 and  $j \neq \langle \lambda, \rho \rangle$ , then

$$L_\lambda \overset{L}{\otimes} \mathfrak{B} \overset{L}{\otimes} \dot{P}_{\mathbf{i}}^{\kappa_j} \cong 0,$$

since  $\mathfrak{B} \overset{L}{\otimes} \dot{P}_{\mathbf{i}}^{\kappa_j}$  is of the form  $\mathfrak{F}_i(\mathfrak{B} \overset{L}{\otimes} \dot{P}_{\mathbf{i}'}^{\kappa_j})$  for a shorter sequence  $\mathbf{i}'$ , and thus, has a projective resolution in which  $P_{\mathbf{i}}^{\kappa(\lambda, \rho)}$  never appears, and thus is killed by tensor product with  $L_\lambda$ .

Thus, we have that as a vector space

$$\mathbb{B}L_\lambda \cong L_\lambda \otimes^L \mathfrak{B} \otimes^L \dot{P}_{\mathbf{i}_\lambda}^{\kappa(\lambda, \rho)} \cong L_\lambda \otimes^L \dot{M}_\lambda(-\langle \lambda, \lambda \rangle) \cong \mathbb{k}[-2\rho^\vee(\lambda)](-2\langle \lambda, \rho \rangle - \langle \lambda, \lambda \rangle).$$

Thus, as a  $E^{\lambda^*, \lambda}$  representation,  $\mathbb{B}L_\lambda$  must be simple, and thus

$$\mathbb{B}L_\lambda \cong L_{\lambda^*}[-2\rho^\vee(\lambda)](-\langle \lambda, \lambda \rangle - 2\langle \lambda, \rho \rangle). \quad \square$$

It also shows more indirectly that  $L_\lambda$  has a beautiful, if more complicated resolution.

**Proposition 2.18.** *There is a resolution*

$$\cdots \longrightarrow M_j \longrightarrow \cdots \longrightarrow M_1 \longrightarrow M_0 \longrightarrow L_\lambda \longrightarrow 0$$

of  $L_\lambda$  with the property that

- $M_{2\rho^\vee(\lambda)-j}$  lies in the subcategory generated by  $S_{\mathbf{i}}^{\kappa_j}$  for all different choices of  $\mathbf{i}$ . In particular, if  $j > \langle \rho, \lambda \rangle$ , then  $M_j = 0$ .
- $M_{2\rho^\vee(\lambda)} \cong M_\lambda(-2\langle \lambda, \rho \rangle)$ .

In particular, both  $\mathbb{K}$  and  $\mathbb{T}$  preserve  $D^\beta$ .

*Proof.* Since we have

$$\text{Ext}^i(S_{\mathbf{i}}^{\kappa_j}, (S_{\mathbf{i}}^{\kappa_k})^*) = 0 \quad \text{if } j \neq k \text{ or } i > 0,$$

the first property is equivalent to showing that

$$\text{Ext}^m(L_\lambda, (S_{\mathbf{i}}^{\kappa_j})^*) = 0 \text{ if } m \neq j.$$

This follows immediately from replacing  $S_{\mathbf{i}}^{\kappa_j}$  by its projective resolution defined in Proposition 2.15.

For the second, we must more carefully analyze this Ext group. By our projective resolution, we have

$$\text{Hom}(M_{2\rho^\vee(\lambda)}, (S_{\mathbf{i}_\lambda}^{\kappa_0})^*) \cong \text{Ext}^{2\rho^\vee(\lambda)}(L_\lambda, (S_{\mathbf{i}_\lambda}^{\kappa_0})^*) \cong \mathbb{k}(2\langle \lambda, \rho \rangle).$$

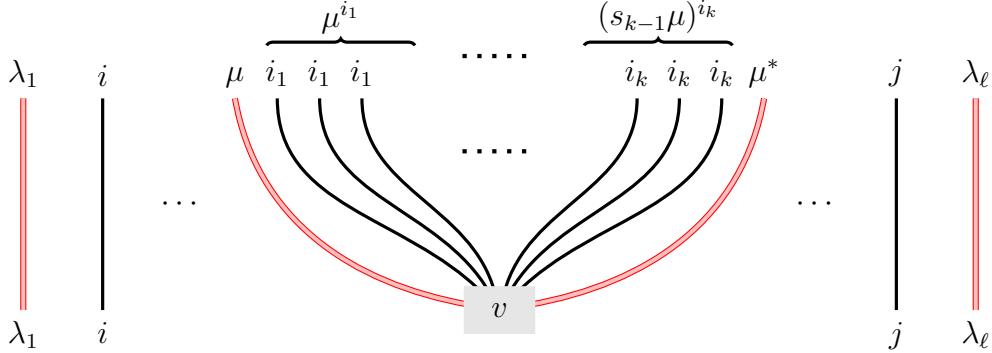
Thus, we must have  $M_{2\rho^\vee(\lambda)} \cong M_\lambda(-2\langle \lambda, \rho \rangle)$ . □

**Corollary 2.19.**  $\text{Ext}^i(L_\lambda, M_\lambda) = \begin{cases} 0 & i \neq 2\rho^\vee(\lambda) \\ \mathbb{k}(2\langle \lambda, \rho \rangle) & i = 2\rho^\vee(\lambda) \end{cases}$ .

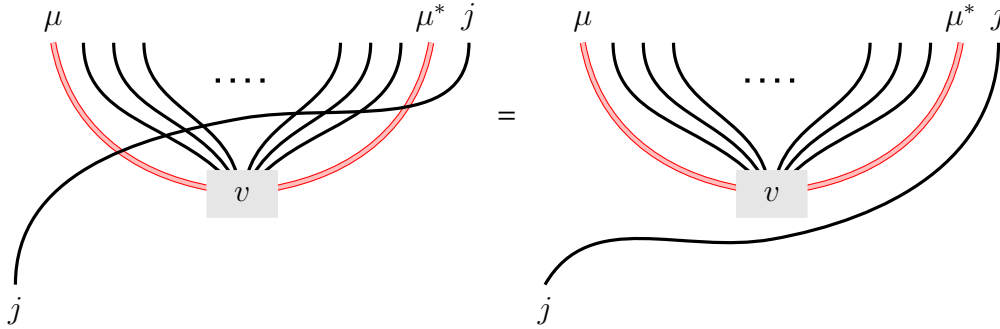
**Corollary 2.20.**  $\text{Ext}^{4\rho^\vee(\lambda)}(L_\lambda, L_\lambda) \cong \mathbb{k}(4\langle \lambda, \rho \rangle)$ .

**2.5. Coevaluation and quantum trace in general.** More generally, whenever we are presented with a sequence  $\lambda$  and a dominant weight  $\mu$ , we wish to have a functor relating the categories  $\lambda$  and  $\lambda^+ = (\lambda_1, \dots, \lambda_{j-1}, \mu, \mu^*, \lambda_j, \dots, \lambda_\ell)$ . This will be given by left tensor product with a particular bimodule.

The coevaluation bimodule  $\mathfrak{K}_\lambda^{\lambda^+}$  is generated by the diagrams of the form



where  $v$  is an element of  $L_\lambda$  and diagrams only involving the strands between  $\mu$  and  $\mu^*$  act in the obvious way, modulo the relation (and its mirror image).



Let  $\mathfrak{F}_i^\kappa$  denote composition of functors where one reads the corresponding idempotent from left to right, and applies  $\mathfrak{F}_i$  when passing a black strand labeled  $i$ , and  $\mathfrak{F}_\lambda$  when passing a red strands labeled  $\lambda$ . This has the useful property that  $\mathfrak{F}_i^\kappa P_\emptyset = P_i^\kappa$ .

We can write  $\lambda = \lambda' \lambda''$  and  $\mathbf{i} = \mathbf{i}' \mathbf{i}''$  as the union of the red/black strands that come before and after the point where  $\mu, \mu^*$  are inserted, with  $\kappa', \kappa''$  be the corresponding  $\kappa$ -functions. Then, we can give an alternate definition of this bimodule by the formula.

$$P_i^\kappa \otimes \mathfrak{K}_\kappa^{\lambda^+} \cong \mathfrak{F}_{\mathbf{i}''}^{\kappa''} (\mathbb{S}^{\lambda'; (\mu, \mu^*)} (P_{\mathbf{i}'}^\kappa \boxtimes L_\mu)).$$

**Definition 2.21.** *The coevaluation functor is the tensor product*

$$\mathbb{K}_\lambda^{\lambda^+} = - \otimes^L \mathfrak{K}_\lambda^{\lambda^+} : \mathcal{V}^\lambda \rightarrow \mathcal{V}^{\lambda^+}.$$

Similarly, the quantum trace functor is the adjoint to this given by

$$\mathbb{T}_\lambda^{\lambda^+} = \text{Ext}(\mathfrak{K}_\lambda^{\lambda^+}, -)(2\langle \lambda, \rho \rangle)[2\rho^\vee(\lambda)] : \mathcal{V}^{\lambda^+} \rightarrow \mathcal{V}^\lambda.$$

Since  $\mathfrak{K}_\lambda^{\lambda^+}$  is projective as a left module, tensor product with it gives an exact functor. The quantum trace functor, however, is very far from being exact.

**Proposition 2.22.**  $\mathbb{K}_\lambda^{\lambda^+}$  categorifies the coevaluation and  $\mathbb{T}_\lambda^{\lambda^+}$  the quantum trace.

*Proof.* We need only prove the former, since the latter follows by adjunction. Furthermore, we may reduce to the case where  $\mu$  is added at the end of the sequence, since all other cases are obtained from this by Hecke operators.

In this case, consider  $\mathbb{K}_{\lambda}^{\lambda^+}(S_i^{\kappa})$ . The resulting module is isomorphic to the standardization

$$\mathbb{S}^{\lambda; \mu, \mu^*}(S_i^{\kappa} \boxtimes L_{\mu})$$

since left-moving crossing the red lines from  $\lambda_m$ 's is trivial since we are considering a standardization and left-moving crossing  $\mu$  is killed since it is positive degree.

This reduces to the case where  $\lambda = \emptyset$ , which we have covered in Propositions 2.12 and 2.13.  $\square$

### 3. KNOT INVARIANTS

**3.1. Constructing knot and tangle invariants.** Now, we will use the functors above to construct tangle invariants.

Using the functors we have presented in the previous section, we can associate a functor  $\Phi(T): \mathcal{V}^{\lambda} \rightarrow \mathcal{V}^{\mu}$  to any diagram of an oriented labeled ribbon tangle  $T$  with the bottom ends given by  $\lambda = \{\lambda_1, \dots, \lambda_{\ell}\}$  and the top ends labeled with  $\mu = \{\mu_1, \dots, \mu_m\}$ .

As usual, we choose a projection of our tangle such that at any height (fixed value of the  $x$ -coordinate) there is a single crossing, single cup or single cap. Thus, we can write our tangle as a composition of these elementary tangles.

For a crossing, we ignore the orientation of the knot, and separate crossings into positive and negative according to the upward orientation we have chosen on  $\mathbb{R}^2$ .

- To a positive crossing, we associate the braiding functor  $\mathbb{B}$ .
- To a negative crossing, we associate its adjoint  $\mathbb{B}^{-1}$  (the left and right adjoints coincide, since  $\mathbb{B}$  is an equivalence).

For the cups and caps, it is necessary to consider the orientation

- To a clockwise oriented cup, we associate the coevaluation.
- To a clockwise oriented cap, we associate the quantum trace.
- We write a counter-clockwise cup/cap using an oppositely oriented one, a crossing, and a ribbon twist, as depicted in Figure 11.

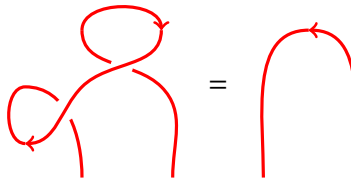


FIGURE 11. Changing the orientation of a cap

Of course, there are two different ways of writing the counterclockwise cup, which are mirror images, and it is not *a priori* obvious that these are isomorphic as functors, but this follows immediately from Proposition 2.17.

**Proposition 3.1.** *The map induced by  $\Phi(T)$  on the Grothendieck groups  $V_\lambda \rightarrow V_\mu$  is that assigned to a ribbon tangle by the structure maps of the category of  $U_q(\mathfrak{g})$ .*

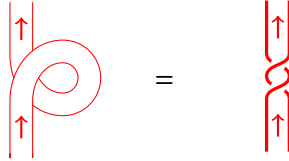
*Proof.* First, we note that since all our functors preserve  $D^\beta$ , these maps are well-defined, since any object in  $\mathfrak{V}^\lambda$  is sent an object of  $\mathfrak{V}^\mu$  which has finite dimensional cohomology in only finitely many degrees, so the alternating sum of the classes of these cohomology groups gives a well-defined element of  $V_\mu = K^0(\mathfrak{V}_\mu)$ .

Of course, we need only check this for each elementary tangle, which was done in Section 2.1, Corollary 2.6 and Proposition 2.22.  $\square$

**Theorem 3.2.** *The functor  $\Phi(T)$  does not depend (up to isomorphism) on the diagram of  $T$ .*

First, we must check that we were justified in rewriting our ribbon twists by Reidemeister 1 moves:

**Lemma 3.3.** *We have an equality (along with all mirror images):*



*Proof.* Since all functors concerned commute with  $\mathfrak{F}_i$  and  $\mathfrak{J}_\lambda$ , we may assume that the pictured red strand is the rightmost, and that no black strands originate to its left. All also commute with standardization functors that add modules on the left, so we may assume that it is the only strand.

Using the standard resolution of  $L_\lambda$  from Proposition 2.18, we have that we have a resolution

$$\cdots \longrightarrow M'_1 \longrightarrow M'_0 \longrightarrow \mathbb{K}_\lambda^{\lambda, \lambda, \lambda^*}(P_\emptyset)$$

where  $M'_{\langle \lambda, \rho \rangle - j}$  is in the subcategory generated by  $S_i^{\kappa'_j}$  where  $\kappa'_j(0) = \kappa'_j(1) = 0$  and  $\kappa'_j(2) = j$ .

Applying  $\mathbb{B}_{1,2}$ , this complex becomes

$$\cdots \longrightarrow \mathbb{B}_{1,2}M'_1 \longrightarrow \mathbb{B}_{1,2}M'_0 \longrightarrow \mathbb{B}_{1,2}\mathbb{K}_\lambda^{\lambda, \lambda, \lambda^*}(P_\emptyset).$$

By Proposition 2.5, we have that  $\mathbb{B}_{1,2}M'_{\langle \lambda, \rho \rangle - j}$  is in the subcategory generated by  $S_i^{\kappa''_j}$  where  $\kappa''_j(1) = \kappa''_j(2) = j$ . In particular, we have that  $\mathbb{B}_{1,2}M_{\langle \lambda, \rho \rangle} \cong M_{\langle \lambda, \rho \rangle}(\langle \lambda, \lambda \rangle)$ .

We note that we have the Ext-vanishing

$$(3.1) \quad \mathbb{T}_\lambda^{\lambda, \lambda, \lambda^*}(S_i^{\kappa''_j}) = 0 \quad \text{unless } j = 0.$$

Thus, it follows that

$$\begin{aligned} \mathbb{T}_\lambda^{\lambda, \lambda^*} \mathbb{B}_{1,2} \mathbb{K}_\lambda^{\lambda, \lambda^*} (P_\emptyset) &\cong \mathbb{T}_\lambda^{\lambda, \lambda^*} M_\lambda(\langle \lambda - 2\rho, \lambda \rangle)[-2\rho^\vee(\lambda)] && \text{(by Thm. 2.18 and (3.1))} \\ &\cong P_\emptyset(\langle \lambda + 2\rho, \lambda \rangle)[2\rho^\vee(\lambda)] && \text{(by Corollary 2.19)} \end{aligned}$$

The result follows. The mirror images follow from similar arguments using either the standard resolution of  $L_\lambda$ , or its dual resolution by costandards.  $\square$

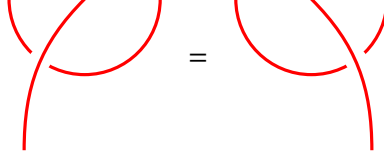


FIGURE 12. The “pitchfork” move

*Proof of Theorem 3.2.* We have already proved all the ribbon Reidemeister moves: RI in Lemma 3.3 above, and RII and RIII as part of Theorem 2.7, so there is only one move of importance left for us to establish: the pitchfork move, shown in Figure 12.

By the same arguments as the proof of Lemma 3.3, we may assume that the pictured red strands are the only ones. We must prove that this move holds for all reflections and orientations. The vertical reflection of the version shown follows for that illustrated by adjunction. We may assume that the cup is clockwise oriented, since the counter clockwise move can be derived from that one using Reidemeister moves II and III. The orientation of the “middle tine” is irrelevant, so we will ignore it.

For the orientation shown above, we need only show this move holds for  $P_\emptyset$  again, since we again have commutation with Hecke functors.

We have two functors  $\mathcal{V}_0^{\lambda, \lambda^*} \rightarrow \mathcal{V}_\mu^{\lambda, \lambda^*}$  given by

$$\Pi_1 = \mathbb{B}_1 \circ \mathbb{S}^{\mu, \lambda + \lambda^*} (P_\emptyset \boxtimes -) \qquad \Pi_2 = \mathbb{B}_2^{-1} \circ \mathbb{S}^{\lambda + \lambda^*, \mu} (- \boxtimes P_\emptyset).$$

**Lemma 3.4.** *The functors  $\Pi_1$  and  $\Pi_2$  coincide.*

*Proof.* First, we multiply both sides by  $\mathbb{B}_2$ , so we must show that we have isomorphisms of functors  $\mathbb{B}_2 \circ \mathbb{B}_1 \circ \mathbb{S}^{\mu, \lambda + \lambda^*} (P_\emptyset \boxtimes -) \cong \mathbb{S}^{\lambda + \lambda^*, \mu} (- \boxtimes P_\emptyset)$ . Since they generate the category, we need only show this isomorphism can be exhibited on the level of projectives.

The left side of the equality is easy to understand, since  $\mathbb{S}^{\lambda + \lambda^*, \mu} (P_\mathfrak{i}^\kappa \boxtimes P_\emptyset) \cong P_\mathfrak{i}^{\kappa^+}$  where  $\kappa^+$  indicates the extension of  $\kappa$  by  $\kappa^+(2) = 2\rho^\vee(\lambda)$ . The isomorphism of this projective to  $\mathbb{B}_2 \circ \mathbb{B}_1 \circ \mathbb{S}^{\mu, \lambda + \lambda^*} (P_\emptyset \boxtimes P_\mathfrak{i}^\kappa)$  is given by sending the generator of the projective to the element where no black strands cross or cross the  $\lambda$  or  $\lambda^*$  strands. We note that this element has degree zero because we are assuming that the roots on the black strands add to  $\lambda + \lambda^*$ .  $\square$

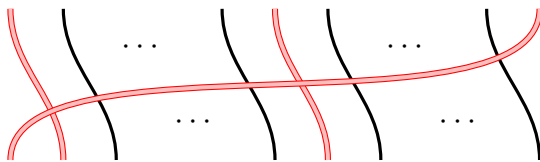


FIGURE 13. The isomorphism of Lemma 3.4

The pitchfork move shown in Figure 12 follows from this lemma, since one side of the move is  $\Pi_1 L_\lambda$  and the other is  $\Pi_2 L_\lambda$ ; the only variation remaining to check is the case where the move is reflected through the page (i.e. with the signs of the crossings given reversed), but this follows from the lemma as well since the two sides are  $(\Pi_1 L_\lambda)^*$  and  $(\Pi_2 L_\lambda)^*$ .  $\square$

Some care must be exercised with the normalization of these invariants, since as we noted in Section 2.1, they are the Reshetikhin-Turaev invariants for a slightly different ribbon element. For 0-framed knots, the change in invariants is quite minor, though.

**Proposition 3.5** ([ST, Theorem 5.21]). *If  $K$  is a 0-framed ribbon knot labeled with  $V_\lambda$ , then the invariants attached to the two different ribbon elements differ by the scalar  $(-1)^{2\rho^\vee(\lambda)}$ .*

Since one of the main reasons for interest in these quantum invariants of knots is their connection to Chern-Simons theory and invariants of 3-manifolds, it is natural to ask:

**Question 3.6.** *Can these invariants glue into a categorification of the Witten-Reshetikhin-Turaev invariants of 3-manifolds?*

**Remark 3.** *The most naive ansatz for categorifying Chern-Simons theory, following the development of Reshetikhin and Turaev [RT91] would associate a category  $\mathcal{C}(\Sigma)$  to each surface  $\Sigma$  which contained an object associated to each isomorphism of  $\Sigma$  with the boundary of a 3-manifold such that the invariants we have given are the Ext-spaces of this object for a knot complement with fixed generating set of  $\mathcal{C}(T^2)$  labeled by the representations of  $\mathfrak{g}$ , and the WRT invariant of a Dehn filling is the Ext space of this object with another associated to the torus filling. While some hints of this structure appear in the constructions of this paper, it's far from clear how they will combine.*

**3.2. Functoriality.** One of the most remarkable properties of Khovanov homology is its functoriality with respect to cobordisms between knots [Jac04]. This property is not only theoretically satisfying but also played an important role in Rasmussen's proof of the unknotting number of torus knots [Ras]. Thus, we certainly hope to find a similar property for our knot homologies. While we cannot present a complete picture at the moment, there are promising signs, which we explain in this section.

The weakest form of functoriality is putting a Frobenius structure on the vector space associated to a circle. This vector space, as we recall, is

$$A_\lambda = \text{Ext}^\bullet(L_\lambda, L_\lambda)[2\rho^\vee(\lambda)](2\langle\lambda, \lambda\rangle).$$

This algebra is naturally bigraded by the homological and internal gradings. The algebra structure on it is that induced by the Yoneda product.

**Proposition 3.7.** *We have a canonical isomorphism*

$$\mathfrak{S}L_\lambda \cong L_\lambda(-4\langle\lambda, \rho\rangle)[-4\rho^\vee(\lambda)].$$

Thus, the functors  $\mathbb{K}$  and  $\mathbb{T}$  are biadjoint up to shift.

In particular,  $\text{Ext}^{4\langle\lambda, \rho\rangle}(L_\lambda, L_\lambda) \cong \text{Hom}(L_\lambda, L_\lambda)^*$ , and the dual of the unit

$$\iota^*: \text{Ext}^{4\langle\lambda, \rho\rangle}(L_\lambda, L_\lambda) \rightarrow \mathbb{k}$$

is a symmetric Frobenius trace on  $A_\lambda$  of degree  $-4\langle\lambda, \rho\rangle$

One should consider this proposition as showing that  $A_\lambda$  satisfies an analogue of Poincaré duality, and thus is a piece of evidence for its relationship to cohomology rings.

*Proof.* The proof of invariance shows that  $\mathfrak{R}L_\lambda \cong L_\lambda$ , as shown pictorially in Figure 14. Thus by Proposition 2.10, we have

$$\mathfrak{S}L_\lambda \cong \mathfrak{S}'L_\lambda = L_\lambda(-2\langle\lambda + \lambda^*, \rho\rangle - \langle 0, 0\rangle)[-2\rho^\vee(\lambda + \lambda^*)] \cong L_\lambda(-4\langle\lambda, \rho\rangle)[-4\rho^\vee(\lambda)]. \quad \square$$

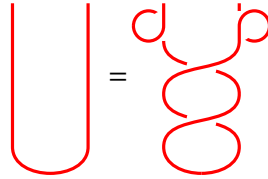


FIGURE 14. The invariance of  $L_\lambda$  under  $\mathfrak{R}$

It would be enough to show that this algebra is commutative to establish the functoriality for flat tangles; we simply use the usual translation between 1+1 dimensional TQFTs and commutative Frobenius algebras (for more details, see the book by Kock [Koc04]).

**Question 3.8.** *Is there another interpretation of the algebra  $A_\lambda$ ? Is it the cohomology (or intersection cohomology) of a space?*

One natural guess, based on the work of Mirković-Vilonen [MV07] and the symplectic duality conjecture of the author and collaborators [BLPWc], is that  $A_\lambda$  is related to the intersection cohomology of the corresponding Schubert cell  $\overline{\text{Gr}}_\lambda$  in the Langlands dual affine Grassmannian.

Another candidate algebra is that induced on  $V_\lambda$  by the quantized “shift of function algebra”  $\mathcal{A}_f$  for a regular nilpotent element  $f$  studied by Feigin, Frenkel, and Rybnikov [FFR].

Following Stroppel [Strb], we can use this biadjunction to give a rather simple prescription for functoriality: for each embedded cobordism in  $I \times S^3$  between knots in  $S^3$ , we can isotope so that the height function is a Morse function, and thus decompose the cobordism into 0-, 1-, and 2-handles. Furthermore, we can choose this so that the projection goes through these handles at times separate from the times it goes through Reidemeister moves. The construct the functoriality map be assigning

- to each Reidemeister move, we associate a fixed isomorphism of the associated functors.
- to the birth of a circle (the attachment of a 2-handle), we associate the unit map  $\mathbb{k} \rightarrow A_\lambda$ .
- to a saddle cobordism (the attachment of a 1-handle), we associate the unit or counit of the biadjunction of quantum trace followed by coevaluation.
- to the death of a circle (the attachment of a 0-handle), we associate the trace  $A_\lambda \rightarrow \mathbb{k}$ .

**Conjecture 3.9.** *This assignment of a map to a cobordism is independent of the choice of Morse function, i.e. this makes the knot homology theory  $\mathcal{K}_\lambda(-)$  functorial.*

#### 4. COMPARISON TO CATEGORY $\mathcal{O}$

**4.1. Cyclotomic degenerate Hecke algebras.** Now, we specialize to the case where  $\mathfrak{g} \cong \mathfrak{sl}_n$ . In this case, we can reinterpret our results in terms of the work of Brundan and Kleshchev [BK08, BK09] who have shown that in this case, the cyclotomic Khovanov-Lauda algebra is a cyclotomic degenerate affine Hecke algebra (cdAHA).

Recall that the degenerate affine Hecke algebra (dAHA) is the algebra with generators  $x_1, \dots, x_d$  and  $w \in S_d$  such that

$$s_i x_j = x_{s_i j} s_i - \delta_{j,i} + \delta_{j,i+1} \qquad x_i x_j = x_j x_i$$

for the simple reflections in  $s_i \in S_d$  and the usual relations between permutations.

We have a natural action of  $H_d$  on the  $\mathfrak{gl}_N$  module  $P \otimes V^{\otimes d}$  for any  $\mathfrak{gl}_n$  representation  $P$ , where  $V = \mathbb{C}^N$  is the defining representation of  $\mathfrak{gl}_N$ . We will be most interested in the case where  $P$  is a certain parabolic Verma module for a parabolic  $\mathfrak{p}$ ; in this case, by the definition of induction,

$$P \otimes V^{\otimes d} \cong U(\mathfrak{gl}_n) \otimes_{U(\mathfrak{p})} (W \otimes V^{\otimes d})$$

for a finite dimensional representation  $W$  of  $\mathfrak{p}$ . This is an object in the parabolic category  $\mathcal{O}$ , which we denote  $\mathcal{O}^{\mathfrak{p}}$ , the full subcategory of  $\mathfrak{gl}_N$ -modules with a weight decomposition where  $\mathfrak{p}$  acts locally finitely.

Attached to each parabolic  $\mathfrak{p} \subset \mathfrak{gl}_{N'}$ , we have a composition  $\pi = (\pi_1, \dots, \pi_\ell)$  such that  $\mathfrak{p}$  is conjugate to block-diagonal matrices for this composition (these can be recovered as the gaps in the finest flag  $\mathfrak{p}$  preserves). These define a weight  $\lambda = \sum_i \omega_{\pi_i}$ ; that is,  $\lambda^j = \#\{i \mid \pi_i = j\}$ .

From here, we can define a quotient of the dAHA called the **cyclotomic dAHA** which is given by

$$H^\lambda = \bigoplus_{d \geq 0} H_d / \left\langle \prod_{i=1}^n (x_1 - i)^{\lambda^i} \right\rangle.$$

This has a natural system of orthogonal idempotents  $e_d$  for all  $d \geq 0$  which project to the image of  $H_d$ . Brundan and Kleshchev show that when  $P$  is the parabolic Verma module associated to the “ground state” tableau on  $\pi$ , then cdAHA acts on  $P \otimes V^{\otimes d}$  via this quotient (though this is not always the whole kernel).

This gives a functor  $\text{Hom}(P \otimes V^{\otimes d}, -) : \mathcal{O}^{\mathfrak{p}} \rightarrow H^\lambda - \text{mod}$ . This functor is very far from being an equivalence, but on each block of  $\mathcal{O}^{\mathfrak{p}}$  it is either 0, or fully faithful on projectives. Thus, certain blocks of  $\mathcal{O}^{\mathfrak{p}}$  can be described in terms of endomorphism rings of modules over  $H^\lambda$ .

In [BK09], Brundan and Kleshchev show that each  $\mathfrak{B}_\mu^\lambda$  is equivalent to a block of  $H^\lambda$ . Thus, using this isomorphism, we can also express  $\mathfrak{B}_\mu^\lambda$  in terms of endomorphisms of modules over  $H^\lambda$ .

There is an idempotent of  $H_d$  associated to any length  $d$  sequence of integers. We let  $e_{\mathfrak{g}}$  be the sum of these idempotents corresponding to sequences of integers in  $[1, n]$ .

**Proposition 4.1** ([BK09]). *We have an isomorphism  $\Upsilon : E^\lambda \rightarrow e_{\mathfrak{g}} H^\lambda e_{\mathfrak{g}} \stackrel{\text{def}}{=} H^{\lambda, n}$  such that  $\Upsilon(y_j e(\mathbf{i})) = e(\mathbf{i})(x_j - i_j)$ .*

**4.2. Comparison of categories.** First, let us endeavor to understand how we can translate our modules  $y_{\mathbf{i}, \kappa} E^\lambda$  into the language of the cdAHA using  $\Upsilon$ . It’s immediate from Proposition 4.1 that

$$\Upsilon(y_{\mathbf{i}, \kappa}) = e(\mathbf{i}) \prod_{j=1}^{\ell} \prod_{k=\kappa(j)+1}^n (x_k - i_k)^{\lambda_j^{i_k}}.$$

However, the strong dependence of this element on  $e(\mathbf{i})$  makes it problematic for use in the Hecke algebra.

We first specialize to the case where  $\lambda_j = \omega_{\pi_j}$  for some  $\pi_j$ . As suggested by the notation, we will later want to think of  $\pi_j$  as a composition. This bit of notation allows us to associate to each  $\kappa$  an element of  $H^{\lambda, n}$  (note that there is no dependence on  $\mathbf{i}$ ):

$$(4.2) \quad z_\kappa = \prod_{j=1}^{\ell} \prod_{k=1}^{\kappa(j)} (x_k - \pi_j)$$

We let  $M_i^\kappa = e(\mathbf{i})z_\kappa H^{\lambda,n}$  and  $M^\kappa = z_\kappa H^{\lambda,n}$ .

**Proposition 4.2.** *For all  $\mathbf{i}$ , we have  $y_{\mathbf{i},\kappa} H^{\lambda,n} = M_i^\kappa$ . In particular, we have an isomorphism  $E^\lambda \cong \text{End}(\bigoplus_\kappa M^\kappa)$ .*

*Proof.* If  $a \neq i_j$ , then we can rewrite  $e(\mathbf{i})$  as

$$e(\mathbf{i}) = (x_j - a)e(\mathbf{i})(-(a - i_j)^{-1} - (a - i_j)^{-2}(x_j - i_j) - (a - i_j)^{-3}(x_j - i_j)^2 - \dots)$$

since  $(x_j - i_j)e(\mathbf{i})$  is nilpotent. It follows that

$$(4.3) \quad e(\mathbf{i})(x_k - \pi_j)H^{\lambda,n} = e(\mathbf{i})(x_k - i_k)\lambda_j^{i_k} H^{\lambda,n}$$

since  $\lambda_j^{i_k} = \delta_{\pi_j, i_k}$ . Thus, applying (4.3) to each term in  $z_\kappa$ , the result follows.  $\square$

We note that the modules  $M^\kappa$  are closely related to the permutation modules discussed by Brundan and Kleshchev in [BK08, §6]. Each way of filling  $\pi$  as a tableau such that the column sums are  $\kappa(i) - \kappa(i - 1)$  results in a permutation module which is a summand of  $M^\kappa$ .

Now we wish to understand how the modules  $M^\kappa$  are related to parabolic category  $\mathcal{O}$ . Let  $N = \sum_j \pi_j$  be the number of boxes in  $\pi$ . As before, the  $\pi_i$  give a composition of  $N$ , and thus a parabolic subgroup  $\mathfrak{p} \subset \mathfrak{gl}_N$ , which is precisely the operators preserving a flag  $V_1 \subset V_2 \subset \dots \subset V$ . If, as usual,  $\kappa$  is a weakly increasing function on  $[1, \ell]$  with non-negative integer values and further  $\kappa(\ell) \leq d$ , then we let

$$V_\kappa^d = V_1^{\otimes \kappa(1)} \otimes V_2^{\otimes \kappa(2) - \kappa(1)} \otimes \dots \otimes V^{d - \kappa(\ell)}$$

as a  $\mathfrak{p}$ -representation. We can induce this representation to an object in  $\mathcal{O}^{\mathfrak{p}}$  which we denote

$$P_\kappa^d \cong U(\mathfrak{gl}_n) \otimes_{U(\mathfrak{p})} (\mathbb{C}_{-\rho} \otimes V_\kappa^d).$$

All the objects  $P_\kappa^d$  live in the subcategory we denote  $\mathcal{O}_{>0}^{\mathfrak{p}}$  which is generated by all parabolic Verma modules whose corresponding tableau has positive integer entries. We also consider a much smaller subcategory which has only finitely many simple objects: let  $\mathcal{O}_n^{\mathfrak{p}}$  be the subcategory of  $\mathcal{O}^{\mathfrak{p}}$  generated by all parabolic Vermas whose corresponding tableau only uses the integers  $[1, n]$ . Let  $\text{pr}_n : \mathcal{O}^{\mathfrak{p}} \rightarrow \mathcal{O}_n^{\mathfrak{p}}$  be the projection to this subcategory ( $\mathcal{O}_n^{\mathfrak{p}}$  is a sum of blocks, so there is a unique projection).

**Proposition 4.3.** *If one ranges over all  $\kappa$  and all integers  $d$ , then  $\bigoplus_{\kappa,d} V_\kappa^d$  is a projective generator for  $\mathcal{O}_{>0}^{\mathfrak{p}}$ .*

*Proof.* This follows from a simple modification of the proof of [BK08, Theorem 4.14]. In the notation of that proof, we have that  $P_d^\kappa \cong R(P_{\kappa(\ell)}^{\kappa^-} \otimes \mathbb{C}_{\pi_\ell}) \otimes V^{\otimes d - \kappa(\ell)}$ , where  $\kappa^-$  is the restriction of  $\kappa$  to  $[1, \ell - 1]$ . As noted in that proof, by induction, this is two functors which preserve projective modules applied to a projective module.

Each of Brundan and Kleshchev's divided power modules is a summand in one of the  $P_d^\kappa$ , as we noted earlier. Since any indecomposable projective of  $\mathcal{O}^{\mathfrak{p}}$  is a summand of a divided power module, the same is true of the  $P_d^\kappa$ 's.  $\square$

**Proposition 4.4.** *For all  $d, \kappa$ , we have*

$$\begin{aligned} z_\kappa H^\lambda e_d &\cong \text{Hom}(P \otimes V^{\otimes d}, P_\kappa^d) \\ M^\kappa e_d &\cong \text{Hom}(P \otimes V^{\otimes d}, \text{pr}_n(P_\kappa^d)). \end{aligned}$$

*Proof.* This rests on a single computation, which is that the image in  $P \otimes V$  of the action of  $\prod_{i=j+1}^\ell (x_1 - \pi_i)$  is

$$U(\mathfrak{gl}_n) \otimes_{U(\mathfrak{p})} (\mathbb{C}_{-\rho} \otimes V_j) \subset U(\mathfrak{gl}_n) \otimes_{U(\mathfrak{p})} (\mathbb{C}_{-\rho} \otimes V) \cong P \otimes V;$$

this follows from [BK08, Lemma 3.3]. This shows that the image of  $z_\kappa$  acting on  $P \otimes V^{\otimes d}$  is  $P_\kappa^d$ , so by the projectivity of  $P \otimes V^{\otimes d}$ , every homomorphism to  $P_\kappa^d$  factors through this one.

We can identify those homomorphisms whose image is in  $\text{pr}_n(P_\kappa^d) \subset P_\kappa^d$  as those killed by some power of  $\chi_j^n = \prod_{i=1}^n (x_j - i)$  for each  $j$  (if a number  $m$  appears in a tableau, then  $x_j - m$  is nilpotent for some  $j$ , and so if  $m \notin [1, n]$ , then  $\chi_j^n$  is invertible for that  $j$ ). Thus, this homomorphism space is the subspace of  $z_\kappa H^\lambda e_d$  on which all  $\chi_j^n$  act nilpotently, which is precisely  $M^\kappa e_d$ .  $\square$

**Corollary 4.5.** *We have an equivalence  $\Xi : \mathfrak{Y}^\lambda \xrightarrow{\cong} \mathcal{O}_n^{\mathfrak{p}}$ .*

We can generalize this statement a bit further: let us now consider the case where the weights  $\lambda_i$  are not fundamental. In this case, to each weight  $\lambda_i$  we have a unique Young diagram given by writing it as a sum of fundamental weights, and we obtain a pyramid  $\pi$  by concatenating these horizontally (this is the pyramid associated earlier to the refinement of  $\lambda$  into fundamental weights). We associate a parabolic  $\mathfrak{p}$  with the pyramid as before.

For each collection of semi-standard<sup>3</sup> tableaux  $T_i$  on each of these diagrams which only use the integers  $[1, n]$ , this gives a tableau on  $\pi$  (now just column-strict). Such tableau can be converted into a module in  $\mathcal{O}^{\mathfrak{p}}$  for  $\mathfrak{gl}_N$  (where  $N = \sum |\lambda_i|$ ) by taking the projective cover of the  $\mathfrak{p}$ -parabolic Verma module corresponding to this tableau. Let  $\mathcal{O}_\lambda^{\mathfrak{p}}$  be the category generated by these projectives.

**Proposition 4.6.** *The functor  $\Xi$  induces an equivalence of  $\mathcal{O}_\lambda^{\mathfrak{p}}$  and  $\mathfrak{Y}^\lambda$ .*

*Proof.* What is clear from Corollary 4.5 is that  $\mathfrak{Y}^\lambda$  is equivalent to the subcategory of  $\mathcal{O}_\lambda^{\mathfrak{p}}$  generated by projectives  $\text{pr}_n(P_\kappa^d)$  for the sequence of weights obtained by breaking  $\lambda$  into fundamental weights, where we require  $\kappa$  to be constant on the blocks of fundamental weights obtained by breaking up  $\lambda_i$ . In terms of category  $\mathcal{O}$ , we only induce finite-dimensional  $\mathfrak{p}$  vector spaces obtained by tensoring the vector spaces which appear in a particular flag preserved by  $\mathfrak{p}$ , the gaps of which encode the sequence  $\lambda$ .

That is, the indecomposable projectives of  $\mathfrak{Y}^\lambda$  are sent to the indecomposable projectives which appear as summands of these  $\text{pr}_n(P_\kappa^d)$ . Thus these are in bijection, and

<sup>3</sup>In [BK08], these are called “standard.”

there can only be  $\dim V_\lambda$  of the latter. Since there is exactly that number of tableaux which are semi-standard in blocks as described above, we need only show that these occur as summands.

This follows from the relationship between the crystal structure on tableaux and projectives in category  $\mathcal{O}$ . Specifically, since any tableau which is semi-standard in blocks can be obtained from the empty tableau by the operations of attaching a fresh Young diagram filled with the ground state tableau and of applying crystal operators, the argument from [BK08, Corollary 4.6] shows that the projective corresponding to such a tableau is a summand of an appropriate  $P_d^\kappa$ .  $\square$

We note that this shows that our categorification corresponds to that for twice fundamental weights of  $\mathfrak{sl}_n$  recently given by Hill and Sussan [HS].

We note that the constructions of this chapter generalize in an absolutely straightforward way to the affine case by simply replacing the results of Section 3 of [BK09] with Section 4.

We let  $\hat{H}_d$  denote the affine Hecke algebra (not the degenerate one we considered earlier). Fix a  $n$ th root of unity  $\zeta$ . The **cyclotomic affine Hecke algebra** or **Ariki-Koike algebra** (introduced in [AK94]) for  $\lambda$  is the quotient

$$\hat{H}^\lambda = \oplus_d \hat{H}_d / \langle (x_1 - \zeta^i)^{\alpha_i^\vee(\lambda)} \rangle.$$

**Theorem 4.7** ([BK09, Main Theorem]). *There is an isomorphism  $E^\lambda \cong \hat{H}^\lambda$ .*

This symmetric algebra has a natural quasi-hereditary cover, called the cyclotomic  $q$ -Schur algebra, defined by Dipper, James and Mathas [DJM98]. Indecomposable projectives over this algebra are indexed by ordered  $\sum_{i=1}^{n-1} \alpha_i^\vee(\lambda)$ -tuples of partitions.

**Proposition 4.8.** *When  $\mathfrak{g} = \widehat{\mathfrak{sl}}_n$ , then  $\mathfrak{B}^\lambda$  is equivalent to the subcategory of representations of the cyclotomic  $q$ -Schur algebra over  $\hat{H}^\lambda$  generated by certain projective modules:*

- *When all  $\lambda_i$  are fundamental,  $\mathfrak{B}^\lambda$  is the subcategory generated by all projective modules for multipartitions whose constituent partitions have a single row.*
- *For a general  $\lambda$ , as before we associate to this a sequence  $\mu$  where the  $\mu_j$ 's are fundamental, and the indices divided into groups that sum to the  $\lambda_i$ . Then  $\mathfrak{B}^\lambda$  is the subcategory generated by all projective modules for multipartitions whose constituent partitions all have a single row, and are only nonempty for the final element in each of groups described above.*

Thus, our categorification can be seen a generalization of the Ariki categorification theorem [Ari96].

**Question 4.9.** *Is there a similar diagrammatic algebra which is Morita equivalent to the entire cyclotomic  $q$ -Schur algebra?*

**4.3. Comparison of functors.** Now, we compare the functors we have described on our categories  $\mathcal{V}^\lambda$  and those on  $\mathcal{O}^p$ . For simplicity, in this section we will assume that  $\lambda$  is a sequence of fundamental weights.

The category  $\mathcal{O}^p$  has a natural endofunctor given by tensoring with  $V$ . Restricting to  $\mathcal{O}_n^p$ , we can take the functor  $f_\bullet = \text{pr}_n(- \otimes V)$ . This functor has a natural decomposition  $f_\bullet = \bigoplus_{i=1}^n f_i$  in terms of the generalized eigenspaces of  $x_1$  acting on  $- \otimes V$ ; we need only take  $i \in [0, n]$  since these are the only eigenvalues of  $x_1$  on the projection to  $\mathcal{O}_n^p$ .

**Proposition 4.10.** *We have a commutative diagram*

$$\begin{array}{ccc} \mathcal{O}_n^p & \xrightarrow{f_i} & \mathcal{O}_n^p \\ \Xi \uparrow & & \uparrow \Xi \\ \mathfrak{Y}^\lambda & \xrightarrow{\mathfrak{F}_i} & \mathfrak{Y}^\lambda \end{array}$$

*Proof.* The functor  $f_\bullet$  corresponds to tensoring a  $H_d^{\lambda, n}$ -module with  $H_{d+1}^{\lambda, n}$ , which corresponds to all ways of going from  $d$  black strands to  $d+1$ , that is the functor  $\bigoplus_{i=1}^n \mathfrak{F}_i$ . Via Brundan and Kleshchev's isomorphism,  $x_n$  acts on  $\mathfrak{F}_i M$  for any  $M$  by  $y_n + i$ ; that is,  $x_n - i$  acts invertibly on  $\mathfrak{F}_j M$  for  $j \neq i$  and nilpotently on  $\mathfrak{F}_i M$ . This shows the desired isomorphism.  $\square$

For any parabolic subalgebra  $\mathfrak{q} \supset \mathfrak{p}$  with Levi  $\mathfrak{l} = \mathfrak{q}/\text{rad } \mathfrak{q}$ , we have an induction functor  $\text{ind}_{\mathfrak{l}}^{\mathfrak{gl}_N} \stackrel{\text{def}}{=} U(\mathfrak{gl}_N) \otimes_{U(\mathfrak{q})} - : \mathcal{O}^p(\mathfrak{l}) \rightarrow \mathcal{O}^p$  where  $\mathcal{O}^p(\mathfrak{l})$  denotes the parabolic category  $\mathcal{O}$  for  $\mathfrak{l}$  and the parabolic  $\mathfrak{p}/\text{rad } \mathfrak{q}$  (here  $\mathfrak{l}$ -representations are considered as  $\mathfrak{q}$  representations by pullback). Let  $\Xi_{\mathfrak{l}}$  be the analogous functor for  $\mathfrak{l}$ .

Choices of  $\mathfrak{q}$  are in bijection with partitions of  $\lambda$  into consecutive blocks  $\lambda_1, \dots, \lambda_k$ .

**Proposition 4.11.** *We have a commutative diagram*

$$\begin{array}{ccc} \mathcal{O}_n^p(\mathfrak{l}) & \xrightarrow{\text{ind}_{\mathfrak{l}}^{\mathfrak{gl}_N}} & \mathcal{O}_n^p \\ \Xi_{\mathfrak{l}} \uparrow & & \uparrow \Xi \\ \mathfrak{Y}^{\lambda_1; \dots; \lambda_k} & \xrightarrow{\mathfrak{S}^{\lambda_1, \dots, \lambda_k}} & \mathfrak{Y}^\lambda \end{array}$$

*Proof.* We need only check this on projectives: consider a parabolic representation of  $\mathfrak{l}$  given by an exterior product  $P = P_1 \boxtimes \dots \boxtimes P_k$ . Then the induction  $\text{ind}_{\mathfrak{l}}^{\mathfrak{gl}_N} P$  is the quotient of the projective  $P'$  corresponding to the concatenation  $T$  of the tableaux  $T_i$

for the  $P_1$  by the image of all maps from projectives higher than  $T$  in Bruhat order through a series of transpositions which change the content of at one of the  $T_i$ .

Similarly, the standardization  $\mathbb{S}^{\lambda_1, \dots, \lambda_k}(\Xi^{-1}(P))$  is the quotient of  $\Xi^{-1}(P')$  by the image of all maps from projectives that correspond to idempotents for sequences where at least one black strand has been moved left from one block to the other. Thus, these functors agree on the level of projective objects.

They agree on morphisms between projectives because of the compatibility of the isomorphism  $\Upsilon$  with the natural inclusions  $H_{d'} \otimes H_{d''} \hookrightarrow H_{d'+d''}$  and  $R(\nu') \otimes R(\nu'') \hookrightarrow R(\nu' + \nu'')$ .  $\square$

Now, let us consider the braiding functors. Associated to each permutation of  $N$  letters, we have a derived twisting functor  $T_w: D^\beta(\mathcal{O}) \rightarrow D^\beta(\mathcal{O})$  (see [AS03] for more details and the definition). Associated to each permutation  $v$  of the weights  $\lambda_i$ , we also have a permutation  $w_v$  of  $N$  letters, which is most easily defined as the permutation induced on the column readings of  $\pi$  and  $v \cdot \pi$ , where  $v \cdot \pi$  is the pyramid produced by concatenating the diagrams of the weights in their permuted order. Alternatively, it is the permutation on diagonal entries induced by permuting the blocks of the matrix representation of  $\mathfrak{p}$  to those of  $\mathfrak{p}_v$ , the parabolic associated to  $v \cdot \pi$ .

From the definition of twisting functors [AS03], we see that  $T_{w_v}$  induces an equivalence of  $D^\beta(\mathcal{O}^{\mathfrak{p}})$  to  $D^\beta(\mathcal{O}^{\mathfrak{p}_v})$ .

**Proposition 4.12.** *We have a commutative diagram*

$$\begin{array}{ccc}
 D^\beta(\mathcal{O}_n^{\mathfrak{p}}) & \xrightarrow{T_{w_v}} & D^\beta(\mathcal{O}_n^{\mathfrak{p}_v}) \\
 \uparrow \Xi & & \uparrow \Xi \\
 \mathcal{V}^\lambda & \xrightarrow{\mathbb{B}_v} & \mathcal{V}^{v \cdot \lambda}
 \end{array}$$

*Proof.* We note that both functors  $T_{w_v}$  commute with translation functors by [AS03, Lemma 2.1(5)]. The same holds for  $\Xi \circ \mathbb{B}_v \circ \Xi$  by Proposition 4.10.

So as usual, we need only compute their behavior on projective parabolic Verma modules on the level of objects. Furthermore, both send parabolic Verma modules to their mutations by a particular change of order. For  $T_{w_v}$  the mutation is that associated to the action of  $v$  on tableaux, and for  $\Xi \circ \mathbb{B}_v \circ \Xi$ , it is given by using  $v$  to reorder the vector  $\alpha$  given sum of the roots that appear between the red lines. These coincide, so the functors are the same.  $\square$

Finally, we turn to describing the functors associated to cups and caps. Recall that we have the equivalence  $\zeta: \mathcal{O}_n^{\mathfrak{p}} \rightarrow \mathcal{O}_n^{\mathfrak{p}'}$  of Enright and Shelton (this is based on the paper [ES87, chapter 11], but developed more fully by Bernstein, Frenkel and

Khovanov in [BFK99]; also, we use the Koszul dual to their equivalence), which says that if  $\pi$  has any columns of height  $n$ , then any block of category  $\mathcal{O}_n^{\mathfrak{p}}$  is equivalent to the block of category  $\mathcal{O}_n^{\mathfrak{p}'}$  associated to  $\pi'$ , the diagram  $\pi$  with that column of height  $n$  removed, where we fill the content of the tableaux is that of the original block with the multiplicity of each number in  $[1, n]$ . The effect of this functor on the simples, projectives and Vermas is simply removing that row (which by column strictness must be the numbers  $[1, n]$  in order).

We also have (co)Zuckerman functors, which are the derived functors of sending a module in  $\mathcal{O}$  to its largest quotient (submodule) which is locally finite for  $\mathfrak{p}$ . These are left (right) adjoint to the forgetful functor  $\mathcal{O}^{\mathfrak{p}} \rightarrow \mathcal{O}$ .

Begin with a pyramid  $\pi$ , and assume  $\pi'$  is obtained from  $\pi$  by replacing a pair of consecutive columns whose lengths add up to  $n$  (a pair of consecutive dual representations in the sequence  $\lambda$ ), with one of length  $n$ , and  $\pi''$  is obtained by deleting them altogether. Then we have a functor  $K : \mathcal{O}^{\pi''} \rightarrow \mathcal{O}^{\pi}$  given by composing the inverse of the Enright-Shelton equivalence for  $\pi''$  and  $\pi'$  with the forgetful functor from  $\mathcal{O}^{\pi'}$  to  $\mathcal{O}^{\pi}$  (which corresponds to an inclusion of parabolic subgroups), and a functor  $T : \mathcal{O}^{\pi} \rightarrow \mathcal{O}^{\pi''}$  which is given by the coZuckerman functor from  $\mathcal{O}^{\pi}$  to  $\mathcal{O}^{\pi'}$  composed with the Enright-Shelton functor  $\zeta : \mathcal{O}^{\pi'} \rightarrow \mathcal{O}^{\pi''}$ .

**Proposition 4.13.** *Both squares in the diagram below commute.*

$$\begin{array}{ccc}
 & \xrightarrow{K} & \\
 D^{\beta}(\mathcal{O}_n^{\mathfrak{p}'}) & & D^{\beta}(\mathcal{O}_n^{\mathfrak{p}}) \\
 & \xleftarrow{T} & \\
 \Xi \uparrow & & \uparrow \Xi \\
 \mathcal{V}^{\lambda} & \xrightarrow{\mathbb{K}} & \mathcal{V}^{\lambda^+} \\
 & \xleftarrow{\mathbb{T}} & 
 \end{array}$$

*Proof.* Obviously, we need only check this for  $K$ , since in both cases, the functors above are in adjoint pairs.

In the case where  $\pi$  has 2 columns and  $N = n$  (so  $\pi'' = \emptyset$ ), then this is clear, since  $K$  sends  $\mathbb{k}$  to the simple for the tableau which places the integers  $[\pi_2 + 1, n]$  in the first column, and  $[1, \pi_2]$  in the second. This is sent under  $\Xi$  to the simple  $L_{\lambda}$ . All other cases follow from this one, using the compatibility results for functors proved in Proposition 4.10 and 4.11.  $\square$

These propositions show that our work matches with that of Sussan [Sus07] and Mazorchuk-Stroppel [MSa], though the latter paper is “Koszul dual” to our approach above. In particular, based on the calculations with foams in [MSa, §7], it follows that

**Proposition 4.14.** *Our invariant  $\mathcal{K}$  coincides with Khovanov homology when  $\mathfrak{g} = \mathfrak{sl}_2$  and Khovanov-Rozansky homology when  $\mathfrak{g} = \mathfrak{sl}_3$ , and in both cases, all components are labeled with the defining representation.*

Obviously, we believe strongly that this homology agrees with that of Khovanov-Rozansky when one uses the defining representation for all  $n$  (this is conjectured in [MSa]), but actually proving this requires an improvement in the state of understanding of the relationship between the foam model of Mackaay, Stošić and Vaz [MSV09] and the model we have presented.

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