Knot homology and quiver varieties

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Reshetikhin-Turaev invariants

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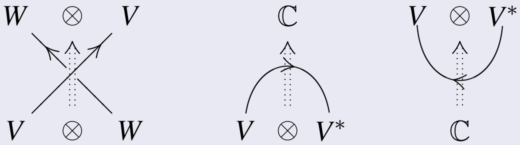

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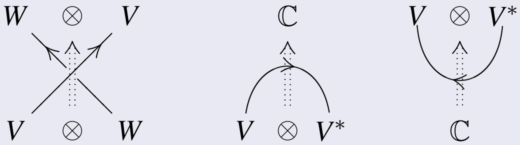

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Composing these together for a given link results in a scalar: the Reshetikhin-Turaev invariant for that labelling.

A little background

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Problem 2: In few particular cases, we have many ways of doing this, but don't know the relationship between them. Even the best current techniques only work for miniscule representations, and the fullest picture has only worked out for the standard representation of $\mathfrak{sl}_n$.

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- One strand, (which includes the work of Khovanov, Stroppel, Sussan, W.-Williamson, etc.) is based around category $\mathcal{O}$, especially parabolic.
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These are constructions of the *same* knot invariants, but there is no obvious correspondence between them. What I'd like to do in this talk is suggest (in varying levels of sketchiness) how to generalize both of these.

Quiver varieties

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Theorem (Nakajima)

Associated to weights $\boldsymbol{\lambda} = \{\lambda_1, \ldots, \lambda_m\}$ with $\lambda = \sum_i \lambda_i$, we have a natural subvariety $\mathfrak{X}_\mu^{\boldsymbol{\lambda}} \subset \mathfrak{Q}_\mu^\lambda$ such that

$$\oplus_\mu H_{top}^{BM}(\mathfrak{X}_\mu^{\boldsymbol{\lambda}}) \cong V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_n}$$

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What about the braiding? More on that later.

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There is a natural category of A^{λ}_{μ}-modules $\mathcal{O}^{\boldsymbol{\lambda}}_{\mu}$ endowed with a grading which categorifies the representation theory coming from quiver varieties; that is,

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Conjecture

The bimodules which give the quantum group action generate a 2-category categorifying $U_q(\mathfrak{g})$, equivalent to that defined by Khovanov and Lauda.

The Fukaya side of the picture

Morally, at least, this story has a symplectic interpretation.

Conjecture

We have a full and faithful inclusion $\mathsf{Fuk}(\mathfrak{Q}^{\lambda}_{\mu}) \hookrightarrow D^b(A^{\lambda}_{\mu} - \mathsf{mod})_{\chi}$ *where* χ *is a particular central character of* A^{λ}_{μ}*.*

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2. then the hard part: an analogue of the theorem of Nadler-Zaslow relating $\mathsf{Fuk}(T^*X)$ to D-modules on X which applies in this situation.

The affine Grassmannian

Assume that I is Dynkin. Then we have an affine Grassmannian for the group G associated to I: $\mathsf{Gr} = G((t))/G[[t]]$, with $G[[t]]$-orbits $\mathsf{Gr}_\lambda = G_\lambda/G[[t]]$ indexed by weights of $\mathfrak{g}_I$. We have $\mathsf{Gr}_\mu \subset \overline{\mathsf{Gr}_\lambda}$ if $(V_\lambda)_\mu \neq 0$.

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We also have a notion of the convolution of two orbits, and a multiplication map m which adds the corresponding coweights

$$\mathsf{Gr}_\lambda \star \mathsf{Gr}_\nu = G_\lambda \times_{G[[t]]} \mathsf{Gr}_\mu \xrightarrow{m} \mathsf{Gr}_{\lambda+\mu}.$$

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Let $\mathfrak{W}^\lambda_\mu$ be the slice to $x_\mu \in \mathsf{Gr}_\mu$ in the larger subvariety $\overline{\mathsf{Gr}_\lambda}$. Let $\tilde{\mathfrak{W}}^{\boldsymbol{\lambda}}_\mu$ be the preimage of $\mathfrak{W}^\lambda_\mu$ under the map $m : \mathsf{Gr}_{\lambda_1} \star \cdots \star \mathsf{Gr}_{\lambda_m} \to \mathsf{Gr}_\lambda$. Let

$$\mathfrak{Y}^{\boldsymbol{\lambda}}_\mu = \{y \in \mathfrak{W}^\lambda_\mu | \lim_{t\to\infty} \check{\rho}(t) \cdot y \in m^{-1}(x_\mu)\}$$

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Proposition (Mirković-Vilonen)

We have an isomorphism $\oplus_\mu H_{top}^{BM}(\mathfrak{Y}_\mu^{\boldsymbol{\lambda}}) = V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_n}$.

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For each $\boldsymbol{\lambda}$, there is a natural category $\mathcal{Q}^{\boldsymbol{\lambda}}_{\mu}$ of modules over B^{λ}_{μ} (given by a definition analogous to the BGG category $\mathcal{O}$) and endowed with a grading such that $K^0(\mathcal{Q}^{\boldsymbol{\lambda}}_{\mu}) \cong V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_n}$, and the action of the braiding is given by tensoring with certain special bimodules.

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To those of you who like the symplectic picture, you can think of this as a replacement for $\mathsf{Fuk}(\tilde{\mathcal{W}}^{\boldsymbol{\lambda}}_\mu)$, which is typically not smooth, and the bimodules as replacements for the monodromy functors.

Recent work of Kamnitzer suggests some correspondences which these bimodules quantize, and that (co)evaluation might be understandable in terms of some vanishing cycle functors.

Strange duality

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However, based on the examples we know, we have conjectured some relationships these varieties should have; one is the isomorphism $H_{top}^{BM}(\mathfrak{X}_\mu^{\boldsymbol{\lambda}}) \cong H_{top}^{BM}(\mathfrak{Y}_\mu^{\boldsymbol{\lambda}})$, but there is a categorifed version of this:

Conjecture (BLPW)

The categories $\mathcal{O}_\mu^{\boldsymbol{\lambda}}$ and $\mathcal{Q}_\mu^{\boldsymbol{\lambda}}$ are Koszul dual. In particular, we have a natural derived equivalence $D^b(\mathcal{O}_\mu^{\boldsymbol{\lambda}}) \cong D^b(\mathcal{Q}_\mu^{\boldsymbol{\lambda}})$.

Knot invariants

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- On the "Grassmannian side," it matches the results of Seidel-Smith and Manolescu for the standard representation of $\mathfrak{sl}_n$.

Avoiding duality

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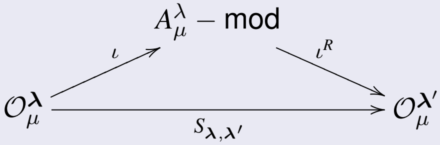
Conjecture

The functor giving the braiding $S_{\boldsymbol{\lambda},\boldsymbol{\lambda}'} : \mathcal{O}_\mu^{\boldsymbol{\lambda}} \to \mathcal{O}_\mu^{\boldsymbol{\lambda}'}$ *is given by the composition of the inclusion* ι *and its* **right adjoint** ι^R.



Thanks, y'all.