

# Geometry of Soergel Bimodules

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## Outline

### Soergel bimodules

- Algebraic part
- Geometric part

### Knot homology and Hochschild homology

### Equivariant formality and equivariant cohomology

- The geometry of Hochschild homology
- Smooth Bruhat cells
- Bott-Samelson spaces

## References:

This slide show can be downloaded from

<http://math.berkeley.edu/~bwebste/HH-SB.pdf>

Some references:

- ▶ W. Soergel, *Kategorie O, Perverse Garben Und Moduln Uber Den Koinvarianten Zur Weylgruppe.*
- ▶ W. Soergel, *The combinatorics of Harish-Chandra bimodules.*
- ▶ M. Khovanov, *Triply-graded link homology and Hochschild homology of Soergel bimodules.*
- ▶ J. Bernstein and V. Lunts, *Equivariant sheaves and functors.*

## Soergel bimodules

Let  $R = \mathbb{C}[x_1, \dots, x_n]/(x_1 + \dots + x_n)$ , and  $s_i$  be the map permuting  $x_i$  and  $x_{i+1}$  and let  $G = \text{SL}(n, \mathbb{C})$ .

Like so many objects in mathematics, Soergel bimodules have a number of definitions:

1. One which explains why anyone ever cared: *projectives in  $\tilde{\mathcal{O}}$*
2. One which is hands-on but totally unilluminating: *tensor products*
3. One which involves disgusting levels of machinery, but which ultimately is the best for working with: *perverse sheaves*

While intimidating at first, a multiplicity of definitions is, in fact, a strength rather than a weakness, allowing us to our problems translate back and forth at will.

## Soergel bimodules for $n = 2$

When  $n = 2$ , then

$$R = \mathbb{C}[x_1, x_2]/(x_1 + x_2) \cong \mathbb{C}[y]$$

with the action of  $s_1$  sending  $y \mapsto -y$ . Thus,  $R^{s_1} = \mathbb{C}[y^2]$  and

$$R_1 \cong R \otimes_{R^{s_1}} R \cong \mathbb{C}[y \otimes 1, 1 \otimes y] \cdot r / (y^2 \otimes 1 - 1 \otimes y^2)$$

### Proposition

The elements  $r \otimes r$  and  $r \cdot y \otimes r = r \otimes y \cdot r$  generate  $R_1 \otimes_R R_1$  as an  $S$ -bimodule, and generate two summands, so  $R_1 \otimes_R R_1 \cong R_1 \oplus R_1\{2\}$ .

### Corollary

Every indecomposable Soergel bimodule for  $n = 2$  is isomorphic to  $R$  or  $R_1$ .

## Soergel bimodules for $n = 3$

When  $n = 3$ , similar calculations show

### Proposition

Every indecomposable Soergel bimodule for  $n = 2$  is isomorphic to one of  $R, R_1, R_2, R_1 \otimes_S R_2, R_2 \otimes_S R_1$  or  $R \otimes_{R^{S_3}} R$ .

Anyone used to playing with  $\text{SL}(3)$  will probably note that we have an obvious bijection from  $S_3$  to the set of indecomposable Soergel bimodules:

$$\begin{array}{lll}
 1 \leftrightarrow R & (12) \leftrightarrow R_1 & (23) \leftrightarrow R_2 \\
 (123) \leftrightarrow R_2 \otimes_R R_1 & (132) \leftrightarrow R_1 \otimes_R R_2 & (13) \leftrightarrow R \otimes_{R^{S_3}} R
 \end{array}$$

## Indecomposable Soergel bimodules

### Question

In general, is the set of indecomposable Soergel bimodules in bijection with  $S_n$ ?

Definition 2 is perfectly useless at answering this sort of question. But from the perspectives of Definitions 1 or 3, it borders on obvious:

### Proposition

Every indecomposable Soergel bimodule is of the form

$$R_w = IH_{B \times B}^*(\overline{BwB}) = \mathbb{H}_{B \times B}^*(\mathbf{IC}_w),$$

for  $w \in S_n$  (and these are all not isomorphic).

## The Hecke algebra

Most algebraic operations one would like to do with bimodules have an equivalent on the geometric side.

For example,

$$\mathbb{H}_{B \times B}^*(\mathbf{IC}_w) \otimes_R \mathbb{H}_{B \times B}^*(\mathbf{IC}_v) \cong \mathbb{H}_{B \times B}^*(\mathbf{IC}_w \star \mathbf{IC}_v)$$

where  $\star$  is convolution product on  $D_{B \times B}^b(G)$ .

Thus, if we construct the split Grothendieck ring of  $\mathbf{Soe}_n$ : the ring over  $\mathbb{Z}[\nu, \nu^{-1}]$  generated by  $[R]$  ( $R \in \text{Ob}(\mathbf{Soe}_n)$ ) and relations

$$[R'] + [R''] \cong [R' \oplus R''] \quad [R'] \cdot [R''] = [R' \otimes_R R''] \quad \nu \cdot [R] = [R\{1\}]$$

it will be isomorphic the convolution algebra of  $B \times B$ -equivariant sheaves on  $G$ , also known as “the Hecke algebra,”  $\mathcal{H}_n$ .

That is,  $\mathbf{Soe}_n$  is a **categorification** of  $\mathcal{H}_n$ .

## The Rouquier complex

One very interesting aspect of the theory of Hecke algebras is the existence of a homomorphism from the braid group  $\varphi : \mathbb{B}_n \rightarrow \mathcal{H}_n$ , defined by  $\varphi(\sigma_i) = T_{s_i}$ . Can this homomorphism be categorified?

Define the complexes of bimodules

$$F(\sigma_i) = R_{s_i}\{-2\} \xrightarrow{1} R_e\{-2\} \quad F(\sigma_i^{-1}) = R_e\{2\} \xrightarrow{x_i \otimes 1 - 1 \otimes x_{i+1}} R_{s_i}$$

### Theorem (Rouquier)

The complex  $F(\sigma) = F(\sigma_{i_1}) \otimes_R \cdots \otimes_R F(\sigma_{i_n})$  depends (up to homotopy) only on  $\sigma = \sigma_{i_1} \cdots \sigma_{i_n} \in \mathbb{B}_n$ . In particular,

$$F(\sigma') \otimes_R F(\sigma'') \simeq F(\sigma'\sigma'').$$

Also,  $[F(\sigma)] = \varphi(\sigma)$ , that is,  $F$  categorifies  $\varphi$ .

## Equivariant cohomology

Let  $EG$  be a contractible space on which  $G$  acts freely.

### Definition

The equivariant cohomology  $H_G^*(X) = \mathbb{H}_G^*(\mathbb{C}_X)$  of a  $G$ -space  $X$  is the cohomology of the Borel space  $EG \times_G X$ . In particular,  $H_G^*(pt) \cong H^*(BG)$  where  $BG = EG/G$ .

There is a variant of equivariant cohomology called **equivariant intersection cohomology**  $IH_G^*(X) = \mathbb{H}_G^*(\mathbf{IC}_X)$  which is better suited for singular spaces, but is the same as  $H_G^*(X)$  for smooth spaces.

We have a map  $EG \times_G X \rightarrow BG$ , giving us an action of  $H_G^*(pt)$  on  $IH_G^*(X)$ .

### Proposition

We have a natural isomorphism  $H_B^*(pt) \cong R$ , so  $H_{B \times B}^*(pt) \cong R \otimes R$ . This geometric action makes  $R_w$  into an  $R - R$ -bimodule.

## Kazhdan-Lusztig theory

The algebra  $\mathcal{H}_n$  has a more familiar presentation, in terms of a basis  $\{T_w\}_{w \in S_n}$ , with relations.

$$T_w T_{s_i} = T_{ws_i} \quad (w < ws_i) \quad T_w T_{s_i} = (\nu - \nu^{-1})T_w + T_{ws_i} \quad (w > ws_i)$$

Of course, we already know a second basis of  $\mathcal{H}_n$ : that provided by the classes  $[R_w]$ . Some of the audience will probably have already deduced the identity of this basis.

### Proposition

The basis  $[R_w]$  is (up to shift) the celebrated basis on Kazhdan and Lusztig.

Anyone in the audience unfamiliar with this basis can take the proposition as a definition, though the K-L basis also has a purely combinatorial definition.

## Triply-graded knot homology

Of course, the homomorphism  $\varphi : \mathbb{B}_n \rightarrow \mathcal{H}_n$  is most interesting because of its applications in knot theory.

### Proposition

There is a natural trace  $\tau : \mathcal{H}_n \rightarrow \mathbb{Z}[\nu^\pm, t^\pm]$ , such that  $P_\sigma(\nu, t) = \tau(\varphi(\sigma))$  is the HOMFLYPT polynomial of the knot  $\bar{\sigma}$  (up to a power of  $q$  and of  $t$ ).

Like  $\varphi$ , this trace has a categorification given by a series of functors called **Hochschild homology**  $HH^i(-)$ , such that  $\tau([R_w])$  is the graded dimension of  $HH^i(R_w)$ .

### Proposition (Khovanov)

The homology of the bigraded complex  $HH^i(F(\sigma))$  is the triply graded knot homology of Khovanov and Rozansky (up to grading shift).

## Hochschild homology

Hochschild homology is the derived functor of  $HH^0(M) = M/[R, M]$  where as usual

$$[R, M] = R \cdot \{r \cdot m - m \cdot r \mid r \in R, m \in M\} \cdot R \subset M.$$

### Definition

For any projective resolution of  $M$ ,

$$\mathbf{P}^\bullet = \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow 0$$

$HH^*(M)$  is the homology of the complex  $HH^0(\mathbf{P}^\bullet)$ .

Note that  $HH^*(R_w)$  has the obvious ‘‘Hochschild’’ grading (which is independent of any grading on  $R$ ) and another ‘‘polynomial’’ grading which arises from a graded projective resolution of  $R_w$ .

## Reduction to compact form

So, now we get use algebraic topology to understand this Hochschild homology. Let  $K = \text{SU}(n)$  be a compact form of  $G$ , and let  $K_w = G_w \cap K$ .

### Proposition

By Iwasawa decomposition,  $IH_T(K_w) \cong IH_B(G_w)$ .

While this is entirely formal, it is useful because we can now use theorems which apply to torus actions on compact spaces. For example, let  $Q$  be the fraction field of  $R$ , and note that  $K_w^T = T$ . By the localization theorem of Goresky, Macpherson and Kottwitz

### Proposition

If  $G_w$  is smooth, then

$$\dim_Q Q \otimes_R HH^*(R_w) = \dim_Q Q \otimes_R H_T^*(K_w) \cong \dim_{\mathbb{C}} H^*(T) = 2^{n-1}.$$

## Gradings

In the case where  $K_w$  is smooth and equivariantly formal, we can separate the Hochschild and polynomial gradings. First, note that we can do this for the trivial module.

By the K nneth theorem,  $H_T^*(T) \cong S \otimes_{\mathbb{C}} H^*(T)$ , so we can write the usual grading as a sum of ‘‘equivariant’’ and ‘‘topological’’ gradings.

### Proposition

The isomorphism  $H_T^*(T) \cong HH^*(R)$  takes the ‘‘topological’’ to the ‘‘Hochschild’’ grading and the ‘‘equivariant’’ to the ‘‘polynomial’’ minus twice the ‘‘Hochschild.’’

If we define a bigrading on  $H_T^*(K_w)$  by the inclusion into  $H_T^*(T)$ , this proposition holds for all  $w \in S_n$  with  $K_w$  smooth and equivariantly formal.

## The geometry of Hochschild homology

Hochschild homology can be interpreted geometrically using Bernstein and Lunts’s equivalence between the equivariant derived category  $D_G(pt)$  with dg-modules over  $H_G^*(pt)$ .

While the general story can get complex, we get a rather simple answer for Soergel bimodules.

### Proposition (W.-Williamson)

$$HH^*(R_w) \cong IH_B^*(G_w),$$

where  $B$  acts on  $G_w$  by conjugation. This isomorphism takes the natural grading on cohomology to the ‘‘polynomial’’ grading on  $HH^*$  minus the ‘‘Hochschild’’ grading.

Keep in mind that if  $G_w$  is smooth, then  $IH_B^*(G_w) \cong H_B^*(G_w)$ .

## Equivariant formality

The localization theorem is most powerful when paired with equivariant formality.

### Definition

We say a  $T$ -space  $X$  is **equivariantly formal** if one of the following equivalent conditions holds

1.  $H_T^*(X)$  is free as a module over  $R$ .
2.  $H_T^*(X) \cong R \otimes_{\mathbb{C}} H^*(X)$ .
3.  $\dim_{\mathbb{C}} H^*(X^T) = \dim_{\mathbb{C}} H^*(X)$ .
4.  $\dim_{\mathbb{C}} H^*(X^T) \geq \dim_{\mathbb{C}} H^*(X)$ .

### Theorem

If  $X$  is equivariantly formal, then the pullback map  $H_T^*(X) \rightarrow H_T^*(X^T)$  is injective.

## The structure of $K_w$

Consider the chain of subgroups

$$\text{SU}(n) \supset \text{SU}(n-1) \supset \cdots \supset \text{SU}(2)$$

Since we have a fiber bundle  $\text{SU}(n-1) \rightarrow \text{SU}(n) \rightarrow S^{2n-1}$ , this makes  $\text{SU}(n)$  into an iterated sphere bundle.

### Proposition

If  $K_w$  is smooth, for some  $w' \in S_{n-1}$  we have a fiber bundle

$$K_{w'}^T \rightarrow K_w \rightarrow S^{2k_w-1}.$$

By induction,  $K_w$  is an iterated sphere bundle, of spheres  $S^{2k_1-1}, \dots, S^{2k_{n-1}-1}$ .

$$\begin{array}{ccccccc} S^{2k_1-1} & S^{2k_2-1} & \cdots & S^{2k_{n-2}-1} & S^{2k_{n-1}-1} \\ \uparrow & \uparrow & & \uparrow & \uparrow \\ K_{w(1)} & \longrightarrow & K_{w(2)} & \longrightarrow & \cdots & \longrightarrow & K_{w(n-2)} & \longrightarrow & K_w \end{array}$$

## Equivariant formality of $K_w$

In fact, the assumption of equivariant formality is redundant.

### Theorem (W.-Williamson)

If  $K_w$  is smooth, it is equivariantly formal.

*Proof.* To apply definition 4 of equivariant formality, we need only show that  $\dim_{\mathbb{C}} H^*(K_w) \leq \dim_{\mathbb{C}} H^*(T) = 2^{n-1}$ . Since  $K_w$  is an  $n - 1$ -times iterated sphere bundle,

$$\dim_{\mathbb{C}} H^*(K_w) \leq \dim_{\mathbb{C}} H^*(S^{2k_1-1}) \cdots \dim_{\mathbb{C}} H^*(S^{2k_{n-1}-1}) = 2^{n-1}.$$

While most of our previous results were true for all compact groups, this result remains a conjecture in types other than  $A_n$ .

## Bott-Samelson Soergel bimodules

Since they appear in the Rouquier complex, we will be interested in the **Bott-Samelson bimodules**

$$R_i \cong R \otimes_{R^{s_{i_1}}} \cdots \otimes_{R^{s_{i_m}}} R \cong R_{s_{i_1}} \otimes_R \cdots \otimes_R R_{s_{i_m}}$$

Let  $P_i$  the parabolic preserving the standard flag minus its  $i$ -dimensional subspace, and let

$$G_i \cong P_{i_1} \times_B P_{i_2} \times_B \cdots \times_B P_{i_m}.$$

This space is smooth, and has a  $B \times B$ -action by left and right multiplication.

### Proposition

For all  $\mathbf{i} = (i_1, \dots, i_m)$ , we have  $R_i \cong H_{B \times B}^*(G_i)$  and  $HH^*(R_i) \cong H_B^*(G_i)$  where, again,  $B$  acts by conjugation.

The differentials of  $F(\sigma)$  can also be realized as pull-backs and pushforwards in cohomology.

## Equivariant formality of $K_i$

Unlike the case of  $K_w$ , our bundle decomposition gives a bound too large to guarantee equivariant formality.

### Question

Is  $K_i$  an equivariantly formal  $T$ -space?

This is a particularly interesting question because

### Proposition

The following are equivalent:

1. The sheaf  $\mathbf{IC}(K_w)$  is equivariantly formal for all  $w \in S_n$ .
2. The space  $K_i$  is equivariantly formal for all words  $\mathbf{i}$ .
3. The space  $K_i$  is equivariantly formal for reduced words  $\mathbf{i}$ .

## The Hochschild homology of smooth Soergel bimodules

This result allows us to directly describe  $HH^i(R_w)$  when  $K_w$  is smooth.

### Theorem

Assume  $K_w$  is smooth,

$$HH^*(R_w) \cong R \otimes_{\mathbb{C}} \wedge^{\bullet}(z_1, \dots, z_{n-1})$$

The element  $z_j$  has bigrading  $(1, 2k_j)$  where  $K_w$  is an iterated bundle of the spheres  $S^{2k_j-1}$ . That is,  $HH^i(R_w)$  is free as an  $R$  module, with a generator labelled by each  $i$ -tuple  $1 \leq j_1 < \cdots < j_i \leq n$  of degree  $\sum_{\ell=1}^i 2k_{j_\ell}$ .

It is worth noting that  $w \in S_n$  with  $K_w$  smooth are characterized combinatorially by pattern avoidance, and that the degrees  $k_i$  can be calculated in terms of the Bruhat conditions for  $w$ .

## The structure of $K_i$

### Proposition

$K_i$  is a torus bundle over an  $m$ -times iterated  $\mathbb{P}^1$  bundle.

### Proposition

The fixed points of the conjugation  $T$  action on  $K_i$  is the subset

$$K_i^T = \bigsqcup_{\epsilon_i \in \{0,1\}; s_1^{\epsilon_1} \cdots s_m^{\epsilon_m} = e} T \cdot (s_1^{\epsilon_1}, \dots, s_m^{\epsilon_m})$$

So  $H^*(K_i^T)$  is just a number of copies of  $H^*(T)$ .

The fact that some difficult combinatorics show up here should clue us in that things will not be so easy here.

### Question

What is  $H^*(K_i)$ ? In particular, what is  $\dim_{\mathbb{C}} H^*(K_i)$ ?

## What's left

Summary: our model

- ▶ gives a uniform description of the Hochschild homology of indecomposable and Bott-Samelson Soergel bimodules.
- ▶ allows us to compute certain cases, as well as leverage for understanding general properties of this homology.

What we hope for is

- ▶ a better understanding of the Bott-Samelson space.
- ▶ a better understanding of non-smooth cases.
- ▶ geometric methods for finding simplifications of the Rouquier complex, and more generally, applications to knot theory.