Representation theory and a strange duality for symplectic varieties

Ben Webster
(joint with Tom Braden, Tony Licata, and Nick Proudfoot)

MIT

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Outline

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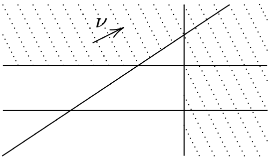
Hyperplane arrangements

Let's start with a little notation.

A **polarized arrangement** $\mathcal{V} = (V, \xi, \nu)$ is

1. A subspace $V \subset \mathbb{R}^n$.
2. An element $\xi \in \mathbb{R}^n/V$ (a coset $V + \xi \subset \mathbb{R}^n$).
3. An element $\nu \in V^*$ (a direction in V).



We'll always assume that this choice is generic.

This picture above is of $V + \xi$. The hyperplanes in the arrangement are the vanishing sets of $t_i|_{V+\xi}$ (where the t_i are the coordinates on $\mathbb{R}^n$).

The **chambers** of $\mathcal{V}$ are the connected components of $(V + \xi) \cap (\mathbb{R}^\times)^n$.

We call a chamber **bounded** if ν achieves a maximum on it. We let $\mathcal{B}$ denote the set of bounded chambers.

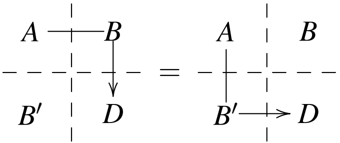
A mysterious algebra

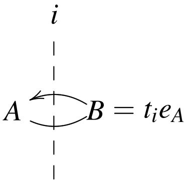
From such an arrangement, one can build an algebra $A(\mathcal{V})$ over $\mathrm{Sym}^\bullet(V^*)$, generated by elements

- c_{AB} for all chambers A, B which are adjacent across a hyperplane.
- idempotents e_A for all chambers A.
- the coordinate functions t_i on $\mathbb{R}^n$, pulled back to V.

with the relations

- $c_{AB}e_{B'} = c_{AB}\delta_B^{B'}$ and $e_{A'}c_{AB} = c_{AB}\delta_A^{A'}$.
- $c_{AB}c_{BD} = c_{AB'}c_{B'D}$.
- $e_A = 0$ if A is not bounded.
- $c_{AB}c_{BA} = t_ie_A$.





Good properties

Despite its mysterious origins, this algebra is quite well behaved.

Definition

An algebra A **quasi-hereditary** *if it has an exceptional collection of* **standard modules** *which generate $A - \mathsf{mod}$ (like Verma modules in category $\mathcal{O}$).*

A positively graded algebra $A = A_0 \oplus A_{>0}$ **Koszul** *if the two natural gradings on $A^\star = \mathrm{Ext}^*_A(A_0, A_0)$ agree. $A^\star$ is called the* **Koszul dual** *of A.*

There's an equivalence of derived categories $D(A - \mathsf{gmod}) \cong D(A^\star - \mathsf{gmod})$.

Theorem (BLPW)

- *$A(\mathcal{V})$ is quasi-hereditary.*
- *$A(\mathcal{V})$ is Koszul.*
- *The center $Z(A(\mathcal{V}))$ is the reduced Stanley-Reisner ring of $\mathbb{R}^n \to V^*$.*

Examples

A few of these are algebras you might have heard of before:

- If
$$V = \{\mathbf{x} \in \mathbb{R}^n | \sum_i x_i = 0\},$$
then the hyperplane arrangement is the faces of a $n-1$-simplex, and the associated category is the block of category $\mathcal{O}$ for $\mathfrak{sl}_n$ including the simple $L_{m\omega_1-\rho}$ (this is also a certain category of representations for the Cherednik algebra of $\mathbb{Z}_n$).
- If $V = \text{span}\,(1,\dots,1)$, then the hyperplane arrangement is n points on a line, and the associated category is a regular block of parabolic category $\mathcal{O}^{\mathfrak{p}}$ for $\mathfrak{sl}_n$, where $\mathfrak{p}$ is the parabolic preserving a line.

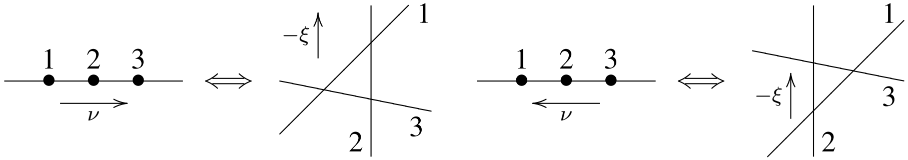
Note: these are Koszul dual!

Gale duality

There's a natural duality on the set of polarized hyperplane arrangements:

$$\mathcal{V} = (V \subset \mathbb{R}^n, \xi, \nu) \iff \mathcal{V}^\vee = (V^\perp \subset \mathbb{R}^n, -\nu, -\xi)$$

This correspondence is surprisingly hard to visualize, so here are some simple examples



Theorem (BLPW)

$(A(\mathcal{V}))^\star \cong A(\mathcal{V}^\vee)$

Derived equivalences

The fact that our result depends on the parameters ξ and ν is a bit dissatisfying. How can we compare the algebras for $\mathcal{V}$ and $\mathcal{V}' = (V, \xi', \nu')$?

Theorem (BLPW)

As along as all parameters are generic, we have an equivalence of derived categories $D(A(\mathcal{V})) \cong D(A(\mathcal{V}'))$, even though the algebras $A(\mathcal{V})$ and $A(\mathcal{V}')$ are generally not Morita equivalent.

These isomorphisms are not canonical at all. In fact, they seem to only be unique up to an action of $\pi_1(\mathsf{Pol}_{\mathbb{C}}(V))$, the complexification of the spaces of choices of polarization of V.

Probably this has something to do with stability conditions on this category or maybe a quotient of it.

Why?

So, these algebras have a really shocking amount of structure for some random relations we wrote down. What could possibly explain this?

If there are any experts in the audience on the Bernstein-Gelfand-Gelfand category $\mathcal{O}_\mathfrak{g}$, you might have noticed that the results above sound an awful lot like ones about $\mathcal{O}_\mathfrak{g}$.

One answer

Both categories can be realized as A-branes on a resolution of a symplectic cone!

- If you're an algebraist: an A-brane is a representation of a deformation quantization of functions on the cone.
- If you're a geometer: an A-brane is an object in the Fukaya category of said resolution.

Symplectic singularities

Let me take a moment to explain what I mean by a symplectic cone:

Definition

We call a quasi-projective variety X **singularly symplectic** *if for any resolution of singularities $\pi : \tilde{X} \to X$, there is a closed algebraic 2-form $\omega \in \Omega^2(\tilde{X})$ such that $\wedge^{top}\omega$ only vanishes on the exceptional set of the resolution.*

We call $\pi : X \to \tilde{X}$ a **symplectic resolution** *if $\wedge^{top}\omega$ is nonvanishing (i.e. X is algebraic symplectic).*

These have recently attracted a lot of attention in recent years, but in a way that's raised more questions than it has answered.

Symplectic cones

We'll be interested in a very special sort of these:

Definition

We call X a **symplectic cone** *if it is singularly symplectic, affine and has a $\mathbb{C}^*$-action ξ which contracts X to a unique fixed point, such that $\xi_t^*\omega = t^m\omega$ for some $m > 0$.*

Equivalently $X = \mathrm{Spec}(A)$ where A is positively graded, and $\{A_r, A_s\} = A_{r+s-m}$ for $m > 0$.

The idea floating in the background of this talk is that one can study X by finding a deformation quantization of A, i.e. an algebra B such that $A = \mathrm{gr}(B)$, where the bracket is the leading order term of commutator in B. More on that later.

Nilpotent cones

Let $\mathcal{N}_{\mathfrak{g}}$ be the cone of nilpotent elements in a complex Lie algebra $\mathfrak{g}$. This gets a singular symplectic structure from the Kostant-Kirillov structure on $\mathfrak{g} \cong \mathfrak{g}^*$.

There's a rather famous symplectic resolution of singularities, the **Springer resolution**

$$\{(n, \mathfrak{b}) | n \in \mathcal{N}, \mathfrak{b} \text{ a Borel}, n \in \mathfrak{b}\} = \tilde{\mathcal{N}} \cong T^*G/B \to \mathcal{N}\}.$$

The universal enveloping algebra of $\mathfrak{g}$ is a deformation quantization of $\mathcal{N}$, so the BGG category $\mathcal{O}$ obviously fits into the algebraic definition of A-branes I gave. For the geometric one, this is trickier, but a theorem:

Theorem (Beilinson-Bernstein, Nadler-Zaslow)

There is in an inclusion $(\mathcal{O}_{\mathfrak{g}})_0 \hookrightarrow \mathrm{Fuk}(T^*G/B)$.

Hypertoric varieties

What symplectic cone corresponds to a hyperplane arrangement? If V is defined over $\mathbb{Z}$, then $\mathbb{C} \otimes_{\mathbb{R}} V^{\perp}$ is the Lie algebra of a subtorus $T \subset (\mathbb{C}^*)^n$.

As always, we have a canonical moment map $\mu : T^*\mathbb{C}^n \to \mathfrak{t}^*$. Let $X /\!\!/_\alpha G$ denote the GIT quotient of a variety X for the character α.

One can do a symplectic reduction in the algebraic category

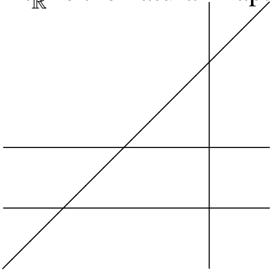
$$\mathfrak{M}_\alpha = \mu^{-1}(0) /\!\!/_\alpha T = \bigsqcup_{v \in V} N^*(T \cdot v) /\!\!/_\alpha T$$

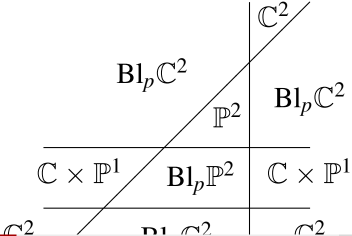
and obtain a **hypertoric variety**, closely tied to the combinatorics of T acting on V. For $\alpha = 0$ this is a symplectic cone, and for α generic, a symplectic resolution of singularities.

You can think of this as an “enhanced cotangent bundle” to the toric variety $\mathbb{C}^n /\!\!/_\alpha T$.

Hypertoric varieties and hyperplane arrangements

Our original data can be recovered as the affine hyperplane arrangement $(\ker \iota, \alpha, -)$ where $\iota : \mathbb{R}^n \to \mathfrak{t}^*_{\mathbb{R}}$ is the natural map.





$A(\mathcal{V})$ and geometry

The category $A(\mathcal{V}) - \mathsf{mod}$ for a polarized arrangement $\mathcal{V} = (V, \xi, \nu)$ has a geometric interpretation similar to that of $\mathcal{O}_{\mathfrak{g}}$.

Conjecture (BLPW)

$A(\mathcal{V}) - \mathsf{mod}$ *has a full and faithful inclusion into either interpretation of A-branes on* $\mathfrak{M}_\xi$*, with its image described by conditions similar to* $(\mathcal{O}_{\mathfrak{g}})_0$.

The algebraic construction is based on an algebra M_V, which we can construct by non-commutative Hamiltonian reduction of the algebra of differential operators $\mathcal{D}_{\mathbb{C}^n}$ by T.

This deformation quantization of $\mathfrak{M}_V$ can be regarded as an analogue of the universal enveloping algebra, and one can search for analogues of all results of Lie theory. But that's another talk.

Canonical deformation

If X is a cone with symplectic resolution of singularities $\tilde{X}$, then

Proposition (Kaledin-Verbitsky)

There is (roughly) a universal deformation $\tilde{Y}$ of $\tilde{X}$ as a symplectic variety over the base $H^2(\tilde{X})$.

- If $\tilde{X}$ is a hyperkähler quotient of T^*V by G, then this is simply given by the family of reductions at different complex moment map values.

$$\tilde{Y} = \bigsqcup_{v \in V} N^*([G,G] \cdot v) /\!\!/_{\alpha} G$$

- If $\tilde{X} = T^*G/B$, then $\tilde{Y} = G \times_B \mathfrak{b}$, the Grothendieck simultaneous resolution.
- If $\tilde{X} = \mathsf{Hilb}^n(\mathbb{C}^2)$, then the fibers of $\tilde{Y}$ are called the Calogero-Moser spaces.

Deformation quantization

Definition

For purposes of this talk, a deformation quantizations of a symplectic cone $\operatorname{Spec} R$ *is a filtered algebra* A *such that* $[A_r, A_s] \subset A_{r+s-m}$*, and* $\operatorname{gr} A \cong R$*, with the induced Poisson structure given by the reduction of commutator.*

There is a generalization of the canonical deformation that includes non-commutative deformations.

Proposition (Bezrukavnikov-Kaledin)

There is a canonical deformation quantization A_Y *of* Y *with center given by* $\mathrm{Sym}^*(H_2\tilde{X})$.

Alternatively, this can be seen as a family A_X^λ *of the deformation quantizations of* X *over* $H^2(\tilde{X})$.

Examples

Some pretty interesting algebras show up when we do this. A couple of them are quite familiar, but it also gives us some new and interesting algebras.

nilcone: $\mathcal{N}_{\mathfrak{g}}$	$\iff$	universal enveloping algebra: $U(\mathfrak{g})$
symmetric power: $\mathrm{Sym}^n(\mathbb{C}^2)$	$\iff$	rational Cherednik algebra U_c for S_n
affine Grassmannian slice: $\mathfrak{W}^\lambda_\mu$	$\overset{?}{\iff}$	primitive quotient of shifted Yangian
quiver variety: $\mathfrak{Q}^\lambda_\mu$	$\iff$	here be dragons
hypertoric variety: $\mathfrak{M}_{\mathcal{A}}$	$\iff$	

"Dragons" is a slight exaggeration; we know what the algebras are, but as far as I know, there is no literature on them.

Category $\mathcal{O}$

For any symplectic cone, we have a class of categories of modules over the deformations quantization, which we can think of as analogues of $\mathcal{O}_{\mathfrak{g}}$.

- If you're an algebraist, you'll take modules locally finite for the action of the non-negative weight subalgebra for this $\mathbb{C}^*$-action.
- If you're a symplectic geometer, you'll take branes with particular "conditions at ∞" determined by the $\mathbb{C}^*$-action.

Conjecture (Too optimistic)

For each symplectic cone with a Hamiltonian $\mathbb{C}^*$-action (with suitable hypotheses), category $\mathcal{O}$ is Koszul, quasi-hereditary, and up to derived equivalence, depends only on the fixed points of the $\mathbb{C}^*$-action.

But what about the Koszul duality results? How can we generalize the relationship between Gale dual hypertoric varieties?

A strange duality

Based on various pieces of evidence, Braden, Licata, Proudfoot and I have suggested that this should reflect some kind of underlying duality between symplectic cones.

Conjecture

This is reflected by a Koszul duality between certain category $\mathcal{O}$'s attached to dual cones.

Observation

Our examples coincide with a notion of duality in physics; they are the Higgs branches of mirror dual 3-dimensional gauge theories.

OK, that's probably not very helpful (I'm not able to convert the physics into a mathematically rigorous definition), so let me give the examples.

Examples of duality

So here's the list of symplectic cones thus far that we believe we have found the dual to:

hypertoric variety: $\mathfrak{M}_{\mathcal{A}}$	$\Longleftrightarrow$	Gale dual: $\mathfrak{M}_{\mathcal{A}^\vee}$
nilcone: $\mathcal{N}_{\mathfrak{g}}$	$\Longleftrightarrow$	Langlands dual: $\mathcal{N}_{{}^L\mathfrak{g}}$
symmetric power: $\mathrm{Sym}^n(\mathbb{C}^2)$	$\Longleftrightarrow$	symmetric power: $\mathrm{Sym}^n(\mathbb{C}^2)$
G_I-instantons on $\widetilde{\mathbb{C}^2/\Gamma_J}$	$\Longleftrightarrow$	G_J-instantons on $\widetilde{\mathbb{C}^2/\Gamma_I}$

$$\Gamma_I \xleftrightarrow{\text{McKay}} G_I$$

quiver variety: $\mathfrak{Q}^\lambda_\mu$	$\Longleftrightarrow$	affine Grass. slice: $\mathfrak{W}^\lambda_\mu$

Simplest interesting example: $T^*\mathbb{P}^{n-1} \Leftrightarrow \widetilde{\mathbb{C}^2/\mathbb{Z}_n}$ or, in terms of cones, $M^{\mathrm{rk}\,1}_{n\times n} \Leftrightarrow \mathbb{C}^2/\mathbb{Z}_n$.

Knot homology

It looks as though this construction might solve a big mystery in the theory of knot homology.

We know how categorify knot invariants attached to the standard representation of $\mathfrak{sl}_n$ in two very different ways,

- using $\mathcal{D}$-modules on partial flag varieties (Stroppel-Mazorchuk, Sussan),
- using the Fukaya category of the resolved slice $\tilde{\mathcal{S}}_\lambda$ to certain nilpotent orbits $GL_n \cdot e_\lambda$ (Seidel-Smith, Manolescu).

The varieties $T^*GL_n/P_{^t\lambda}$ and $\tilde{\mathcal{S}}_\lambda$ are related by symplectic duality.

Thus, symplectic duality gives a general framework that includes this coincidence of knot invariants.

Knot homology

This also suggests a generalization to all Reshetikhin-Turaev invariants.

Conjecture

The monoidal category of integrable representations of a Kac-Moody algebra has a categorification (including the braiding and duality) given by category $\mathcal{O}$'s of quiver varieties (and dually, affine Grassmannians).

- This is exactly what is required to directly categorify Reshetikhin and Turaev's construction of the knot invariants.
- Nakajima gave a description of the quiver variety and $\mathbb{C}^*$-action associated to an (ordered) sequence of weights.
- Zheng described a category of microlocal perverse sheaves which carry a categorified $U_q(\mathfrak{g})$ action, described using Hecke correspondences.
- Recent work of Kamnitzer suggests correspondences which could be used on the affine Grassmannian side to construct the braiding.

Equivariant cohomology of $\tilde{X}$

Assume for now that we have a torus T with isolated fixed points acting on $\tilde{X}$, and $\tilde{X}$ is equivariantly formal. That is, we have an injection

$$H_T^*(\tilde{X}) \to H_T^*(\tilde{X}^T) \cong \oplus_{a \in \tilde{X}^T} H_T(\{a\}).$$

Let $R_{\tilde{X}}$ be the subring of $H_T^*(\tilde{X})$ generated by H_T^2 over H_T^0. More geometrically, we have

$$\operatorname{Spec} R_{\tilde{X}} = \bigcup_{a \in \tilde{X}^T} H_2^T(\{a\}) \subset H_2^T(\tilde{X}).$$

That is, all information about $R_{\tilde{X}}$ is encoded in this subspace arrangement.

There is an obvious "duality" on subspace arrangements, sending all subspaces to their annihilator. Let

$$R_{\tilde{X}}^\vee = \mathbb{C}[\bigcup_{a \in \tilde{X}^T} H_2^T(\{a\})^\perp] \subset H_2^T(\tilde{X})^*.$$

Examples

To keep up with our running examples:

Variety $\tilde{X}$	Spec R	**Duality**
T^*G/B	$\bigcup_{w\in W} \Gamma_w \subset \mathfrak{t}^* \oplus \mathfrak{t}^*$	Langlands
$\mathfrak{M}_{\mathcal{V}}$	$\bigcup_{\beta \text{ a basis of } V^*} \mathbb{C}^\beta \subset \mathbb{C}^n$	Gale
$\mathsf{Hilb}^n(\mathbb{C}^2)$	$\bigcup_{\lambda\vdash n}(1, \mathrm{Con}(\lambda)) \subset \mathbb{C}^2$	self-dual

Observation (Goresky-MacPherson, BLPW)

For all the examples above where the natural torus action has isolated fixed points, the “symplectic dual” $\tilde{X}^\vee$ also has an action of a torus S such that

$$R_{\tilde{X}}^\vee = R_{\tilde{X}^\vee}.$$

GM duality for Koszul algebras

Interestingly, the same phenomenon holds for a general class of Koszul algebras, independent of any connection to geometry.

Any Koszul algebra A over an algebraically closed field k has a canonical flat deformation $\grave{A}$ over $Z(A^\star)_2$ the degree 2 part of the center of the dual $A^\star$.

Assume that A is quasi-hereditary, and the center $Z(\grave{A})$ is also flat.

Let R_A be the subalgebra of $Z(\grave{A})$ generated by $Z(\grave{A})_2$. As before, $\operatorname{Spec} R_A \subset Z(\grave{A}^\star)$ is a union of subspaces. Let $R_A^\vee$ be the coordinate ring of the union of the annihilators.

Theorem (BLPW)

$R_A^\vee = R_{A^\star}$

As a corollary, proving a categorical duality would imply the cohomological duality on the previous page.

Future goals

- Investigate degenerate hypertoric cases, quiver varieties and affine Grassmannians.
    - More generally, find the equivalent of every theorem of Lie theory in these cases.
- Categorification of:
    - the entire theory of hyperplane arrangements.
    - tensor products of $\mathfrak{g}$-representations.
    - Reshetikhin-Turaev invariants.
- Give unified proof of duality statements (at the moment, all depend on combinatorics).
- Find more symplectic cones.
- Physics?

Thanks, y'all.