II. CALCULUS OF VECTORS, DYADICS AND TENSORS

A. Introduction & Review

1. Scalars & Vectors

   - Scalar: magnitude only, e.g., mass, temperature.
   - Vector: characterized by magnitude & direction, represented geometrically as an arrow.

   \[ \vec{A} = \vec{B} \Rightarrow A = B \]

   (Nevertheless, it is important to keep in mind that the effect of a given vector may depend upon its location.)

   **NOTATION:** I will typically indicate a vector quantity by an underline, e.g., \( \vec{a} \) or \( \vec{b} \).

   Another common method is to use arrows, \( \overrightarrow{a} \), \( \overrightarrow{b} \).

2. Cartesian Coordinate System

   a. We will indicate the unit base vectors as:
      \[ \vec{e}_1 = (1, 0, 0), \vec{e}_2 = (0, 1, 0), \vec{e}_3 = (0, 0, 1) \]

   b. In order to describe a vector, you must give both the components and the base vectors.

      \[ \vec{a} = a_x \vec{e}_1 + a_y \vec{e}_2 + a_z \vec{e}_3 \]

   c. Recall the definition of the SCALAR PRODUCT of 2 vectors:

      \[ \vec{a} \cdot \vec{b} = |\vec{a}| \cdot |\vec{b}| \cdot \cos \theta \]

      where \( |\vec{a}| \), \( |\vec{b}| \) are the magnitudes of \( \vec{a} \) and \( \vec{b} \).

      Also, since \( \vec{i} \cdot \vec{i} = 1 \), \( \vec{i} \cdot \vec{j} = 0 \), \( \vec{i} \cdot \vec{k} = 0 \), etc., then

      \[ \vec{a} \cdot \vec{b} = a_x b_x + a_y b_y + a_z b_z \]
c. Scalar product (continued)

(ii) Clearly, we also have \( a \cdot b = b \cdot a \) and \( a \cdot (b + c) = a \cdot b + a \cdot c \) and \( |a|^2 = a \cdot a = a^2 \).

d. VECTOR PRODUCT (also called CROSS PRODUCT)

(i) The vector product of 2 vectors \( a, b \) is defined as

\[
\mathbf{a} \times \mathbf{b} = |a| |b| \sin \theta \mathbf{e} = \mathbf{e} \times b
\]

\( a \times a = 0 \).

**NOTE:**

- My notation for this operation is \( \mathbf{a} \times \mathbf{b} = \mathbf{e} \times b \).
- Where \( e \) is a unit vector in the direction perpendicular to the plane formed by \( a \) & \( b \), as given by the RIGHT-HAND RULE.

(ii) From the definition: \( \mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a} \) and \( \mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c} \)

It also follows that \( \mathbf{i} \times \mathbf{j} = \mathbf{k}, \mathbf{j} \times \mathbf{k} = \mathbf{i}, \mathbf{k} \times \mathbf{i} = \mathbf{j} \) etc.

(iii) You may also remember writing something like

\[
\mathbf{a} \times \mathbf{b} = \begin{vmatrix} i & j & k \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix} = i(a_yb_z - a_zb_y) + j(a_zb_x - a_xb_z) + k(a_xb_y - a_yb_x)
\]

\( \Rightarrow \) Much of the above is cumbersome & frightfully lengthy to write.

We now introduce a special notation which will simplify many manipulations.

B. EINSTEIN INDEX NOTATION AND THE SUMMATION CONVENTION

1. Let us reconsider some of the above. From now on keep in mind that we are representing vectors in a three-dimensional world.

So, we will now label \((x, y, z)\) coordinates by \((1, 2, 3)\).

Let the vector \(a\) have components \(a_i\), base vectors \(e_i\).

Then,

\[
a = a_1 e_1 + a_2 e_2 + a_3 e_3 = \sum_{i=1}^{3} a_i e_i = a_i e_i \quad (= a_j e_j)\]

This idea must be clear in your mind before you move on.

\( \Rightarrow \) From now on, we will not write the summation symbol. Instead we will invoke the SUMMATION CONVENTION — if an index appears twice, we will know that we should do a summation \(\sum_{i=1,2,3}\).
2. Scalar product revisited

Consider two vectors \( \mathbf{a} = a_i \mathbf{e}_i \), \( \mathbf{b} = b_j \mathbf{e}_j \)

Then, \( \mathbf{a} \cdot \mathbf{b} = \sum_{i=1}^{3} a_i b_j \mathbf{e}_i \cdot \mathbf{e}_j = \sum_{i=1}^{3} a_i b_i \) (= \( a_1 b_1 + a_2 b_2 + a_3 b_3 \))

(Do you understand the base vectors are orthogonal? \( \mathbf{e}_i \cdot \mathbf{e}_j = 0 \) if \( i \neq j \), \( \mathbf{e}_i \cdot \mathbf{e}_i = 1 \) if \( i = j \))

where we again invoke the summation convention and drop the summation symbol.

3. Kronecker delta \( \delta_{ij} \) (\( i = 1, 2, 3 \), \( j = 1, 2, 3 \))

a. Definition:

\[
\delta_{ij} = \begin{cases} 
0 & \text{if } i \neq j \\
1 & \text{if } i = j 
\end{cases}
\]

clearly \( \mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij} \)

b. With this shorthand we write

\[
\mathbf{a} \cdot \mathbf{b} = a_i b_j \mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij} \mathbf{a}_i \mathbf{b}_j = a_i b_j \delta_{ij} = a_i b_i
\]

\( \delta_{ij} \) is a replacement operator

In this eqn, \( i \) would be called \( \text{the Summation index} \), and we again remark that a different dummy index was used for each vector \( \mathbf{a}_i, \mathbf{b}_j \).

\( \delta_{ij} \) implies the double sum \( \sum_{i=1}^{3} \sum_{j=1}^{3} a_i b_j \delta_{ij} \)

NOTES: If you like, you may think about \( \delta_{ij} \) as the components of the identity matrix

\[
\begin{pmatrix} 
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
\]

only non-zero if \( i = j \)

c. Remarks:

(i) \( \delta_{ii} = \delta_{11} + \delta_{22} + \delta_{33} = 3 \) using the summation convention \( \sum_{j=1}^{3} \delta_{ij} \)

(ii) Very often, one will not write the unit vectors \( \mathbf{e}_i \)

and will write \( A_i \) where it is understood that \( i \) may be either 1, 2, or 3. In this case \( i \) would be called a free index since it is free to take on the values 1, 2, or 3.

Similarly, the vector eqn \( \mathbf{a} = \mathbf{b} \) may be written

\( a_i \mathbf{e}_i = b_i \mathbf{e}_i \) or \( a_i = b_i \) and

Since \( i \) only appears once on each side of the eqn, it is free to take on the values 1, 2, or 3 so this stands for 3 separate equalities: \( a_1 = b_1 \), \( a_2 = b_2 \), \( a_3 = b_3 \).

Another example:

\( (\mathbf{a} \cdot \mathbf{b}) \mathbf{c} = a_i b_j c_j \mathbf{e}_i \cdot \mathbf{e}_j \) or \( a_i b_j c_j \), \( j \) is free to take on the values 1, 2, or 3.
4. Permutation Symbol \( \varepsilon_{ijk} \) \( i=1,2,3 \quad j=1,2,3 \quad k=1,2,3 \)

4. Definition: \( \varepsilon_{ijk} = \begin{cases} +1 & \text{if } i,j,k \text{ are all different} \\ -1 & \text{if any two indices are the same} \end{cases} \)

In particular,

\[ \varepsilon_{123} = 1 \quad \varepsilon_{312} = -1 \quad \varepsilon_{231} = -1 \quad \varepsilon_{321} = 1 \]

**Note:** By even permutation, we mean that an even # of interchanges of the indices must occur to get back to the order 123; analogous for meaning of odd permuation.

b. This definition has the following cyclic and interchange property:

\[ \varepsilon_{ijk} = \varepsilon_{kij} = \varepsilon_{jki} \]

and if two indices are simply interchanged, the sign changes:

\[ \varepsilon_{ijk} = - \varepsilon_{ijk} \quad \text{or} \quad \varepsilon_{ijk} = - \varepsilon_{ijk} \]

Also, since \( \varepsilon_{ijk} \) can each independently take on the values 1,2,3, then \( \varepsilon_{ijk} \) represents 27 quantities.

c. We also have \( \varepsilon_i \wedge \varepsilon_j = \varepsilon_{ijk} \varepsilon_k \)

and by referring to the figure at right, everything is o.k.:

\[ \varepsilon_1 \wedge \varepsilon_2 = + \varepsilon_3 = \varepsilon_{123} \varepsilon_3 \quad \text{etc.} \]

d. We now have an effective shorthand notation for representing the vector product.

let \( \mathbf{c} = \mathbf{a} \wedge \mathbf{b} \); write \( \mathbf{a} = a_i \varepsilon_i \); \( \mathbf{b} = b_j \varepsilon_j \)

\[ \Rightarrow \mathbf{c} = a_i \varepsilon_i \wedge b_j \varepsilon_j = a; b_j \varepsilon_{ijk} \varepsilon_k \]

\[ \mathbf{a} \wedge \mathbf{b} = a; b_j \varepsilon_{ijk} \varepsilon_k \]

**Note:** Carefully the order of the indices

or with \( \mathbf{c} = c_k \varepsilon_k \), we have \( c_k = a; b_j \varepsilon_{ijk} \varepsilon_k \) \( \text{w.r.t. represents 3 eqns for } k=1,2,3 \).

**Exercise:** Verify that this is in agreement with the 'matrix' definition on pg. 57. Use summation convention on \( i,j,k \).
e. triple scalar product: \( \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) \)

Again we are careful to use different dummy indices for each vector so

\[
\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = a_i \varepsilon_i \cdot \left( b_j \varepsilon_j \times c_k \varepsilon_k \right) = a_i \varepsilon_i \cdot \left( b_j c_k \varepsilon_j \varepsilon_k \right)
\]

\[
= a_i b_j c_k \varepsilon_j \varepsilon_k \delta_{ij}
\]

\[
= \varepsilon_{ijk} a_i b_j c_k = \varepsilon_{ijk} a_i \delta_{jk} c_k = (a^\times b) \cdot c = (c^\times a) \cdot b
\]

by using cyclic property of \( \varepsilon_{ijk} \)

Exercise - convince yourself that these 2 identities follow from index expression.

Recall also that

\[
\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \det \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \varepsilon_{ijk} a_i b_j c_k
\]

index representation of the 3x3 determinant

5. Useful identities involving \( \varepsilon \) and \( \delta \)

\[
\varepsilon_{ijk} \varepsilon_{klm} = \delta_{il} \delta_{jm} - \delta_{im} \delta_{jl}
\]

Proof: verify by brute force for each of the 81 terms! However, it is best to make your life easier by noticing that both sides change sign if either \( i \leftrightarrow j \) or \( l \leftrightarrow m \) are interchanged. Also, both sides vanish if \( i = j \) or \( l = m \). Then, consider remaining terms like:

\[
\varepsilon_{12k} \varepsilon_{kl2} = \varepsilon_{12k} \varepsilon_{k2l} + \varepsilon_{12k} \varepsilon_{k2l} = 1
\]

and

\[
\delta_{i} \delta_{j} - \delta_{j} \delta_{i} = 1 \quad \text{so o.k.}
\]

License

\[
\varepsilon_{12k} \varepsilon_{kl3} = \varepsilon_{12k} \varepsilon_{k3l} + \varepsilon_{12k} \varepsilon_{k3l} = 0 \quad \text{also} \quad \delta_{i} \delta_{j} - \delta_{j} \delta_{i} = 0 \quad \text{so o.k., etc.}
\]

Example 1: show that \( \varepsilon_i = \frac{1}{2} \varepsilon_{mnj} \varepsilon^{m} \varepsilon_{n} \delta_{nj} \delta_{ji} - \delta_{ij} \delta_{nj} \delta_{ni} \)

Well, \( \varepsilon_{mnj} \varepsilon^{m} \varepsilon_{n} = \varepsilon_{mnj} \varepsilon^{mj} \varepsilon^{nj} = \varepsilon_{mnj} \varepsilon^{mj} \varepsilon^{nj} = (2 \delta_{ij} - \delta_{ij}) \varepsilon^{m} = 2 \varepsilon_{i} \times
\]

Example 2: show that \( \mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{b} (\mathbf{a} \cdot \mathbf{c}) - \mathbf{c} (\mathbf{a} \cdot \mathbf{b}) \)

\[
\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = a_i \varepsilon_i \times \left( b_j \varepsilon_j \times c_k \varepsilon_k \right) = a_i \varepsilon_i \times \left( b_j c_k \varepsilon_j \varepsilon_k \right) = a_i b_j c_k \varepsilon_j \varepsilon_k \delta_{ij}
\]

\[
= a_i b_j c_k \varepsilon_{jkm} = a_i b_j c_k \varepsilon_{jkm} = a_i b_j c_k \left( \delta_{jm} \varepsilon_{km} - \delta_{jm} \varepsilon_{km} \right)
\]

\[
= a_i b_j c_k \varepsilon_{kem} - a_i b_j c_k \varepsilon_{kem} = (q \cdot c_i) b_j \varepsilon_{kem} - (q \cdot b_i) c_k \varepsilon_{kem} = (q \cdot c_i) b_j - (q \cdot b_i) c_k \times
\]
Some Examples of the use of index notation

First, a brief summary of the important ideas

(i) \( \delta_{ij} \cdot \delta_{ij} = \delta_{ij} = \begin{cases} 1 & i=j \end{cases} \)

(ii) \( \delta_{ij} \cdot \delta_{ij} = E_{ijk} \cdot \delta_{jk} = \begin{cases} +1 & i,j,k \text{ an even permutation of 1,2,3} \\ -1 & i,j,k \text{ an odd permutation of 1,2,3} \\ 0 & \text{any two indices the same} \end{cases} \)

(iii) Summation convention: whenever a subscript appears twice, a summation from 1 to 3 is implied.

Examples:

(i) \( \delta_{ik} \cdot \delta_{jk} = \delta_{ij} \) since \( \delta_{ij} \) is only nonzero when \( j=k \) so the \( k \) in \( \delta_{ik} \) may be replaced by \( j \).

(ii) \( \delta_{ii} = \delta_{11} + \delta_{22} + \delta_{33} = 1 + 1 + 1 = 3 \) note: Since \( i \) was a dummy index, \( \delta_{ii} = \delta_{kk} = \delta_{mm} \) etc.

(iii) \( \delta_{ij} \cdot E_{ijk} = E_{iik} = 0 \) since two of the indices are the same.

(iv) \( E_{ijk} \cdot E_{ijk} = E_{iik} \cdot E_{kjm} \) by first rotating the indices on the second \( E \);

next, use the identity:

\[ E_{iik} \cdot E_{kjm} = \delta_{ij} \cdot \delta_{jm} = \delta_{ij} \cdot \delta_{jm} = 3 \cdot \delta_{in} - \delta_{in} = 2 \delta_{in} \]

(v) \( a_{m} \cdot b_{n} \cdot E_{mjq} = a_{n} \cdot b_{m} \cdot E_{mjq} \)?

\( \Rightarrow \) new appear twice in each term so summation is implied.

But, \( m \) \& \( n \) are simply dummy variables, i.e., we could just as well use another letter.

So, examine the second term \( a_{m} \cdot b_{n} \cdot E_{mjq} \).

\( a_{m} \cdot b_{n} \cdot E_{mjq} = -a_{n} \cdot b_{m} \cdot E_{mjq} \) j now let \( j = n \), \( n = k \)

\( = -a_{j} \cdot b_{k} \cdot E_{jkq} \) which is the same as the above since summation on \( j,k,k \) is implied.

\( = -a_{m} \cdot b_{n} \cdot E_{mjq} \) by letting \( j = m \), \( k = n \)

So, we see that:

\( a_{m} \cdot b_{n} \cdot E_{mjq} - a_{n} \cdot b_{m} \cdot E_{mjq} = 2 \cdot a_{m} \cdot b_{n} \cdot E_{mjq} \)

\( g^{i} b \) component of \( g^{ab} \)
C. Some vector calculus (taking derivatives of vector functions)

1. Notation: we will use the vector \( \mathbf{x} \) to denote the vector location of a point in space.

\[ \mathbf{x} = (x_1, x_2, x_3) \]

The value of \( \phi \) depends on location in space.

One can discuss scalar fields \( \phi(\mathbf{x}) = \phi(x_1, x_2, x_3) \) or just \( \phi(x_i) \)
and one can discuss vector fields \( \mathbf{q}(\mathbf{x}) = q_1(x_1, x_2, x_3) \mathbf{e}_1 + q_2(x_1, x_2, x_3) \mathbf{e}_2 + q_3(x_1, x_2, x_3) \mathbf{e}_3 \)

Each of the components of the vector \( \mathbf{q} \) depends on \( x_i \) location in space

2. Differentiation of vectors

a. Suppose \( \mathbf{q} = \mathbf{q}(t) = q_i(t) \mathbf{e}_i \)

Then \( \frac{d \mathbf{q}}{dt} = \frac{d q_i(t)}{dt} \mathbf{e}_i \) since the cartesian base vectors \( \mathbf{e}_i \) are constant vectors.

We will now consider spatial derivatives of vectors, e.g.,

\[ \frac{\partial}{\partial x} \mathbf{q}(\mathbf{x}) \text{ or } \frac{\partial}{\partial y} \mathbf{q}(\mathbf{x}) \]

3. Gradient operator — Section 9.3 Greenberg; Sections 6.7, 6.8 Hildebrand

Let \( \phi(\mathbf{x}) \) be a scalar function which varies with position \( x_1, x_2, x_3 \) in space.

The rate of variation of \( \phi \) in the \( x_1 \)-direction is \( \frac{\partial \phi}{\partial x_1} = \frac{\partial \phi}{\partial x_1} \), in the \( x_2 \)-direction is \( \frac{\partial \phi}{\partial x_2} = \frac{\partial \phi}{\partial x_2} \), and in the \( x_3 \)-direction is \( \frac{\partial \phi}{\partial x_3} = \frac{\partial \phi}{\partial x_3} \).

We introduce the vector:

\[ \nabla \phi \equiv \nabla \phi = \mathbf{e}_1 \frac{\partial \phi}{\partial x_1} + \mathbf{e}_2 \frac{\partial \phi}{\partial x_2} + \mathbf{e}_3 \frac{\partial \phi}{\partial x_3} = \mathbf{e}_i \frac{\partial \phi}{\partial x_i} \]

(or gradient of \( \phi \))

Gradient operator \( \nabla() = \mathbf{e}_1 \frac{\partial}{\partial x_1} + \mathbf{e}_2 \frac{\partial}{\partial x_2} + \mathbf{e}_3 \frac{\partial}{\partial x_3} \)

\[ = \mathbf{e}_i \frac{\partial}{\partial x_i} \]

"comma" notation to indicate differentiation with respect to \( x_i \)
3. More about the gradient operator

- Relation between the gradient and the directional derivative

Consider a small displacement $d\gamma$, where $|ds| = ds$.

The unit tangent vector $\hat{r}$ in the direction of $d\gamma$ is $\hat{r} = \frac{d\gamma}{ds}$

Then, the rate-of-change of $\phi$ in the direction of $\hat{r}$ is

$$\hat{r} \cdot \nabla \phi = \frac{d\phi}{ds} = \frac{d\phi}{dx_1} \hat{e}_1 + \frac{d\phi}{dx_2} \hat{e}_2 + \frac{d\phi}{dx_3} \hat{e}_3$$

$$\nabla \phi = \frac{d\phi}{ds} \hat{r} = \frac{d\phi}{dx_1} \hat{e}_1 + \frac{d\phi}{dx_2} \hat{e}_2 + \frac{d\phi}{dx_3} \hat{e}_3$$

- Directional derivative of $\phi$ in the $\hat{r}$ direction

Now, consider a surface $\phi(x) = \text{constant}$

Clearly $d\phi = 0$ for any displacement along the surface, so $\mathbf{v} \cdot \nabla \phi = 0 < c$

Since $\hat{r}$ is a tangent vector to the surface, it follows that $\mathbf{v} \cdot \nabla \phi = 0$,

$\nabla \phi$ is a vector perpendicular to the surface $\phi = \text{constant}$

$\nabla \phi$ is a vector normal to the surface $\phi = \text{constant}$

4. Divergence of a vector field $\nabla \cdot \mathbf{f}$ or $\text{div} \, \mathbf{f}$

- Simply compute using standard rules.

$$\nabla \cdot \mathbf{f} = \left( \hat{e}_i \cdot \frac{\partial f_j}{\partial x_i} \right) = \hat{e}_i \cdot \frac{\partial f_j}{\partial x_i} + f_j \hat{e}_i \cdot \frac{\partial \hat{e}_j}{\partial x_i}$$

Using the product rule

$$= \delta_{ij} \frac{\partial f_j}{\partial x_i}$$

Sometimes used $0$ since the $\hat{e}_i$'s do not vary with position in space.

NOTE: Now that you have gone through this, make your life easier.
The $\hat{e}_i$'s are constant vectors with respect to differentiation so we know it is ok to simply write

$$\nabla \cdot \mathbf{f} = \hat{e}_i \frac{\partial f_j}{\partial x_i} \cdot (f_j \hat{e}_i) = \hat{e}_i \cdot \frac{\partial f_j}{\partial x_i} = \frac{\partial f_j}{\partial x_i}$$

Also, whenever you see a term like $\frac{\partial f_j}{\partial x_i}$, you now know $\frac{\partial f_j}{\partial x_i} = \nabla \cdot \mathbf{f}$. 

A. An identity using index notation:

\( \nabla \cdot (\phi \mathbf{f}) = \epsilon_{i} \frac{\partial}{\partial x_{i}} \left( \phi f_{j} \epsilon_{j} \right) \)

(i) use product rule
(ii) \( \epsilon_{j} \) are constant vectors
(iii) the inner product (\( \cdot \))
only operates on vectors,
not the scalar components \( \phi f_{j} \)

\[ = \delta_{ij} \left[ \frac{\partial \phi}{\partial x_{i}} f_{j} + \phi \frac{\partial f_{j}}{\partial x_{i}} \right] \]

\[ = \frac{\partial \phi}{\partial x_{j}} f_{j} + \phi \frac{\partial f_{j}}{\partial x_{j}} = (\nabla \phi) \cdot \mathbf{f} + \phi \nabla \cdot \mathbf{f} \]

\( \nabla \cdot (\phi \mathbf{f}) = (\nabla \phi) \cdot \mathbf{f} + \phi \nabla \cdot \mathbf{f} \)

Notice how similar this is to the normal product rule of differentiation.

C. Interpretation of the Divergence of a Vector Field

Recall the Divergence Theorem which relates certain volume integrals to integrals over a bounding surface:

\[ \iiint_{V} \nabla \cdot \mathbf{f} \, dV = \oiint_{S} \mathbf{f} \cdot \mathbf{n} \, dS \]

\( \nabla \cdot \mathbf{f} \)

In the field of fluid dynamics, we find a very nice physical interpretation of the divergence of a vector field.

Consider the flow of a fluid of constant density (e.g., water - such a fluid is called incompressible).

Let \( \mathbf{v}(\mathbf{x}) \) be the velocity of the fluid at a pt \( \mathbf{x} \).

Let \( S \) be some fixed boundary drawn in the fluid. The net flow rate through a surface with differential area \( dS \) is \( \left( \mathbf{v} \cdot \mathbf{n} \right) dS \).

The total flow through the surface is found by integrating over \( S \):

\[ \oint_{S} \mathbf{v} \cdot \mathbf{n} \, dS = \text{in} - \text{out} = 0. \]

And since this must be true for any choice of the volume element \( V \) we conclude \( \nabla \cdot \mathbf{v} = 0 \) for all \( \mathbf{x} \).

For an incompressible fluid, the vanishing of the divergence of the velocity field is associated with conservation of mass. Alternatively, if there is a source (or sink) of mass, \( \nabla \cdot \mathbf{v} \) measures net flux away from a point.
4. Curl of a vector field \( \nabla \times \mathbf{f} \) or curl \( \mathbf{f} \)

a. Again, simply compute using standard ideas

\[
\nabla \times \mathbf{f} = e_i \frac{\partial}{\partial x_i} \times (f_j e_j) = (e_i \times e_j) \frac{\partial f_j}{\partial x_i}
\]

\[
\nabla \times \mathbf{f} = e_{ijk} \frac{\partial f_j}{\partial x_i} e_k
\]

\( \leftarrow \) As before, the \( e_i \) are constant vectors and the curl \( (\times) \) operation only affects vectors

Sometimes people will write this as

\[
(\nabla \times \mathbf{f})_k = e_{ijk} \frac{\partial f_j}{\partial x_i}
\]

\( \uparrow \) indicates the \( k \)th component of the vector \( \nabla \times \mathbf{f} \)

b. Alternatively, just go through and show that the above agrees with what you have seen in earlier vector calculus courses.

First,

\[
\nabla \times \mathbf{f} = (e_i \times e_j) \frac{\partial f_j}{\partial x_i} \quad \text{and since the summation convention has been assumed and the variables } i,j \text{ appear twice, we must sum } i=1 \rightarrow 3, \ j=1 \rightarrow 3.
\]

or

\[
\nabla \times \mathbf{f} = (e_i \times e_j) \frac{\partial f_j}{\partial x_i} + (e_i \times e_2) \frac{\partial f_2}{\partial x_i} + (e_i \times e_3) \frac{\partial f_3}{\partial x_i} = e_3 (\frac{\partial f_3}{\partial x_2} - \frac{\partial f_2}{\partial x_3}) + e_2 (\frac{\partial f_2}{\partial x_3} - \frac{\partial f_3}{\partial x_1}) + e_1 (\frac{\partial f_1}{\partial x_3} - \frac{\partial f_3}{\partial x_2})
\]

\[
= e_1 \begin{vmatrix} e_1 & e_2 & e_3 \\ \frac{\partial f_3}{\partial x_2} & \frac{\partial f_2}{\partial x_3} & \frac{\partial f_1}{\partial x_3} \\ f_1 & f_2 & f_3 \end{vmatrix}
\]

which is probably how you previously saw it represented.

c. Another identity:

\[
\nabla \times \nabla \phi = e_i \frac{\partial}{\partial x_i} \times (e_j \frac{\partial \phi}{\partial x_j}) = e^e e_j \frac{\partial^2 \phi}{\partial x_i \partial x_j} = e_{ijk} \frac{\partial^2 \phi}{\partial x_i \partial x_j} e_k
\]

But notice that by using the properties of \( e_{ijk} \),

\[
e_{ijk} \frac{\partial^2 \phi}{\partial x_i \partial x_j} = -e_{jik} \frac{\partial^2 \phi}{\partial x_j \partial x_i} = -e_{jik} \frac{\partial^2 \phi}{\partial x_j \partial x_i} = 0 \quad \text{by comparing with the first term, relabeled } i \rightarrow j \text{ twice continuously differentiable}
\]

\[
\therefore \nabla^2 \phi = 0 \quad \text{for any scalar function } \phi
\]
4. Curl of a vector field (continued)

\[
\nabla \cdot (a \times b) = \varepsilon_{ijk} \frac{\partial}{\partial x_i} (a_j b_k) e_j e_k
\]

\[
= \varepsilon_{ijk} \frac{\partial}{\partial x_i} (a_j b_k) e_j e_k = \varepsilon_{ijk} \frac{\partial}{\partial x_i} (a_j b_k) e_j e_k
\]

\[
= \varepsilon_{ijk} \frac{\partial}{\partial x_i} (a_j b_k) e_j e_k = \varepsilon_{ijk} \frac{\partial}{\partial x_i} (a_j b_k) e_j e_k
\]

\[
\Rightarrow \frac{\partial}{\partial x_i} (a_j b_k) e_j e_k = \frac{\partial}{\partial x_i} (a_j b_k) e_j e_k
\]

or in vector notation.

\[
\nabla \cdot (a \times b) = (\nabla a) \cdot b - (\nabla b) \cdot a
\]

5. Interpretation of the curl of a vector field - Again, use the velocity field of a fluid flow as an example.

Let \( \mathbf{v}(x) \) be the velocity of a fluid flow. We will now see that \( \nabla \times \mathbf{v} \) provides a measure of the average angular velocity.

Consider 2 line segments in the flow; examine planar motions for simplicity.

For small rotations, \( \tan \alpha \sim \alpha \) and \( \tan \beta \sim \beta \) so that in a short time \( \Delta t \)

\[
\alpha \sim \tan \alpha = \left[ \frac{v_2(x_1 + \Delta x_1) - v_2(x_1)}{\Delta x_1} \right] \Delta t \sim \frac{\partial v_2}{\partial x_1} \Delta t
\]

and

\[
\beta \sim \tan \beta = \left[ \frac{v_1(x_2 + \Delta x_2) - v_1(x_2)}{\Delta x_2} \right] \Delta t \sim \frac{\partial v_1}{\partial x_2} \Delta t
\]

\[
\Rightarrow \text{average rate of counterclockwise rotation of fluid particle about the } x_3 \text{-axis is } \frac{1}{2} \left[ \frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2} \right] = \frac{1}{2} (\nabla \times \mathbf{v})_3
\]

And, in general, the average rate of rotation of a fluid particle about the \( x_1 \)-axis is

\[
\frac{1}{2} (\nabla \times \mathbf{v})_1
\]
D. Integral Theorems

1. Divergence Theorem (or Gauss' Theorem)

This theorem relates integrals over volumes to integrals over the bounding surface(s).

The theorem states that given a continuous vector function \( \mathbf{f} \) with continuous first partial derivatives, then

\[
\text{DIVERGENCE THEOREM} \quad \iiint_V \nabla \cdot \mathbf{f} \, dV = \iiint_S \mathbf{n} \cdot \mathbf{f} \, dS
\]

where \( \mathbf{n} \) is the unit outward normal from \( V \) using index notation, we write

\[
\mathbf{n} \cdot \mathbf{f} = (n_1 f_1 + n_2 f_2 + n_3 f_3) \quad \text{on \( S \)}
\]

(Recall the proof of the theorem from Math21)

Written out in 3D:

\[
\iiint_V \left( \frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} + \frac{\partial f_3}{\partial x_3} \right) \, dV = \iiint_S \left( n_1 f_1 + n_2 f_2 + n_3 f_3 \right) \, dS
\]

2. Planar versions of the Divergence Theorem

Consider some area \( A \) in the plane bounded by the curve \( C \) and let \( \mathbf{n} \) and \( \mathbf{t} \) be the unit normal and unit tangent vectors along the boundary.

Since \( \mathbf{t} = \frac{d\mathbf{r}}{ds} \rightarrow \mathbf{t} \, ds = dx_1 \, \mathbf{e}_1 + dx_2 \, \mathbf{e}_2 \)

and

\[
\mathbf{n} \cdot \mathbf{t} = 0
\]

So,

\[
\int_{A} \left( \frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} \right) \, dA = \int_{C} \mathbf{n} \cdot \mathbf{f} \, ds = \oint_{C} (f_1 \, dx_2 - f_2 \, dx_1)
\]

Let

\[
f_1 = N(x_1, x_2) \\
f_2 = -M(x_1, x_2)
\]

\[
\int_{A} \left( \frac{\partial N}{\partial x_1} - \frac{\partial M}{\partial x_2} \right) \, dA = \int_{C} (M \, dx_1 + N \, dx_2)
\]

which you may recall seeing in a previous course.
2. Planar versions (continued)

On the other hand, if we begin with the last eqn but identify N, M with the components of a vector as

\[ \mathbf{a}(x) = a_1(x_1, x_2) \mathbf{e}_1 + a_2(x_1, x_2) \mathbf{e}_2 \]

and let \( a_1 = M, \ a_2 = N \), then

\[ \int (a_1 \, dx_1 + a_2 \, dx_2) = \int \left( \frac{\partial a_2}{\partial x_1} - \frac{\partial a_1}{\partial x_2} \right) \, dA \]

\[ \mathbf{C} \]

\[ \int a \cdot \mathbf{t} \, ds = \int \left( \nabla \times \mathbf{a} \right) \cdot \mathbf{e}_3 \, dA \]

\[ \mathbf{C} \]

\[ A \]

which is the planar version of \( \text{Stokes' Theorem} \)

we will come back to \text{Stokes} Theorem shortly.

3. Some Theorems which follow directly from the Divergence Theorem

Begin with

\[ \int \frac{\partial f_i}{\partial x_i} \, dV = \int n_i f_i \, ds \]

\[ \mathbf{V} \]

\[ \mathbf{S} \]

\( i.e. \)

\[ \int \mathbf{V} \cdot f \, dV = \int \mathbf{n} \cdot f \, ds \]

\( \mathbf{V} \)

\( \mathbf{S} \)

(i) let \( \mathbf{f} = \phi \mathbf{b} \) where \( \phi = \phi(x) \), but \( \mathbf{b} \) is an arbitrary constant vector

so we have

\[ \left( \int \mathbf{V} \cdot f \, dV \right) \mathbf{b} = \left( \int \frac{\partial \phi}{\partial x_i} \, dV \right) \mathbf{b}_i = \left( \int n_i \phi \, ds \right) \mathbf{b}_i \]

where \( \mathbf{b}_i \) can be taken as \( \mathbf{b} \) the
dependencies because \( \mathbf{b} \)

is a constant vector.

Since this eqn must be true for arbitrary \( b \), we conclude

\[ \int \frac{\partial \phi}{\partial x_i} \, dV = \int n_i \phi \, ds \]

\( \mathbf{V} \)

\( \mathbf{S} \)

\( i = 1, 2, 3 \)

and this is a vector equality as it holds for each component.

and this is a vector equality as it holds for each component.

which is Gauss' Theorem for a scalar function.
(ii) Another theorem which follows from the Divergence Theorem is obtained by letting

\[ f = \nabla \phi \]

Then since

\[ \nabla \cdot f = \nabla \cdot (\nabla \phi) = \sum_{i=1}^{3} \frac{\partial}{\partial x_i} \left( \sum_{j=1}^{3} \frac{\partial \phi}{\partial x_j} \right) = \frac{\partial^2 \phi}{\partial x_1^2} + \frac{\partial^2 \phi}{\partial x_2^2} + \frac{\partial^2 \phi}{\partial x_3^2} = \nabla^2 \phi \]

we have

\[
\int_V \nabla^2 \phi \, dV = \int_S n \cdot \nabla \phi \, dS
\]

the LAPLACIAN

4. Green's Theorem

(Hildebrand, p 301-2)

(i) Begin with the Divergence Theorem and let \( f = \psi \nabla \phi \)

Then,

\[
\int_S (\psi \nabla \phi) \cdot dS = \int_V \nabla \cdot (\psi \nabla \phi) \, dV = \int_V \sum_{i=1}^{3} \frac{\partial}{\partial x_i} (\psi \frac{\partial \phi}{\partial x_i}) \, dV
\]

\[
= \int_V \frac{\partial}{\partial x_i} (\psi \frac{\partial \phi}{\partial x_i}) \, dV = \int_V (\frac{\partial \psi}{\partial x_i} \frac{\partial \phi}{\partial x_i} + \psi \frac{\partial^2 \phi}{\partial x_i^2}) \, dV
\]

\[
\therefore \int_S \frac{\partial}{\partial \phi_{|n}} \psi \, dS = \int_V \left( -\psi \nabla \cdot \phi + \psi \nabla^2 \phi \right) \, dV
\]

Green's 1st form or Green's 1st Identity

(ii) Interchange \( \psi, \phi \) in the above eqn, then subtract from the eqn just derived.

\[
\int_S \left( \psi \frac{\partial \phi}{\partial n} - \phi \frac{\partial \psi}{\partial n} \right) \, dS = \int_V (\psi \nabla^2 \phi - \phi \nabla^2 \psi) \, dV
\]

Green's 2nd form or Green's Second Identity

Exercise: derive this eqn.
An interesting aside:

(iii) Green's Theorems are often useful for proving some very general results.

For example, begin with Green's first form; let \( U = \phi \)

Then,
\[
\int_S \phi \frac{\partial \phi}{\partial n} \, dS = \int_V \left[ \nabla \phi \cdot \nabla \phi + \phi \nabla^2 \phi \right] \, dV
\]

(a p.d.e.)

\[ \rightarrow \]

Now suppose you wish to solve \( \nabla^2 \phi = 0 \) (Laplace's eqn) in \( V \) subject to the b.c. \( \phi = 0 \) on \( S \)

What is \( \phi(x) \) for \( x \in V \)?

Well, we are given \( \phi = 0 \) on the boundary \( S \) so,
\[
\int_S \phi \frac{\partial \phi}{\partial n} \, ds = 0.
\]

Furthermore, since \( \nabla^2 \phi = 0 \) in \( V \), the eqn above reduces to
\[
\int_V \left[ \nabla \phi \cdot \nabla \phi \right] \, dV = 0
\]

But the integrand \( \nabla \phi \cdot \nabla \phi \) is the sum of squares, so is always positive. For the integral to vanish we must consequently require \( \nabla \phi \cdot \nabla \phi = 0 \) everywhere in \( V \).

\( \nabla \phi = 0 \), i.e., \( \phi \) doesn't vary with spatial position in \( V \).

\[ \Rightarrow \phi = \text{constant in } V \]

But \( \phi = 0 \) on \( S \) and so therefore \( \phi = 0 \) throughout \( V \).
5. A further generalization of the Divergence Theorem:

We began by stating the Divergence Theorem:

\[ \int_V \nabla \cdot \mathbf{F} \, dV = \int_S \mathbf{n} \cdot \mathbf{F} \, dS \]

where \( S \) represents the closed surface enclosing the volume \( V \) and \( \mathbf{n} \) is the unit outward normal from the volume.

We then proved a form suitable for scalar functions,

\[ \int_V \nabla \phi \, dV = \int_S \mathbf{n} \cdot \phi \, dS \]

or using index notation,

\[ \int_V \frac{\partial \phi}{\partial x_i} \, dV \, e_i = \int_S \mathbf{n} \cdot \phi \, dS \, e_i \]

If two vectors are equal, their corresponding components are equal, so

\[ \int_V \frac{\partial \phi}{\partial x_i} \, dV \, e_i = \int_S \mathbf{n} \cdot \phi \, dS \, e_i \quad \text{for } i = 1, 2 \text{ or } 3. \]

b. We now construct a form useful when the cross-product appears.

For example, consider integrals of the form

\[ \int_V \nabla \times \mathbf{F} \, dV \]

It is simplest to work in index notation. Notice that \( \nabla \times \mathbf{F} \) represents a vector so the result of the integration is also a vector.

Basically, we can proceed by considering each component of the vector separately.

So using index notation,

\[ \int_V \nabla \times \mathbf{F} \, dV = \sum \frac{\partial F_k}{\partial x_j} \, e_j \times e_i \, dV = \sum \frac{\partial F_k}{\partial x_j} \, dV \, e_j \times e_i \]

But for each \( k \) and \( i \) we know that

\[ \int \frac{\partial F_k}{\partial x_j} \, dV = \int \mathbf{n} \cdot f_k \, dS \]

In other words, for each \( k, i \) use eqn (e) and integrate \( f_k = \phi \) so that

\[ \int \frac{\partial \phi}{\partial x_j} \, dV \, e_j \times e_i = \int \mathbf{n} \cdot \phi \, dS \, e_j \times e_i \]

Hence

\[ \int_V \nabla \times \mathbf{F} \, dV = \sum \frac{\partial F_k}{\partial x_j} \, dV \, e_j \times e_i = \int_S \mathbf{n} \cdot f_k \, dS \, e_j \times e_i = \int_S \mathbf{n} \times \mathbf{F} \, dS \]

\[ \int_S \mathbf{n} \times \mathbf{F} \, dS \]
C. Notice that we can conveniently summarize all of the above results concerning the Divergence Theorem as follows:

\[ \int \nabla \cdot \Phi \, dV = \int \mathbf{n} \cdot \Phi \, dS \]

where \( \Phi \) is any quantity, scalar or vector, and \( \cdot \) is any operation (scalar product, vector product or a simple gradient operation) that makes sense.

**Examples:**

(i) Let \( \mathbf{a} = \text{constant vector} \)

\[ \int_{\mathcal{S}} \mathbf{n} \cdot \mathbf{a} \, dS = \int_{\mathcal{V}} \nabla \cdot \mathbf{a} \, dV = 0 \]

(ii) Evaluate

\[ \int_{\mathcal{S}} \mathbf{n} \cdot (\nabla f) \, dS \]

By the Divergence Theorem,

\[ \int_{\mathcal{S}} \mathbf{n} \cdot (\nabla f) \, dS = \int_{\mathcal{V}} \nabla \cdot (\nabla f) \, dV = 0. \]

(iii) Evaluate \( \int_{\mathcal{S}} \mathbf{n} \cdot \nabla r^2 \, dS \)

Using index notation,

\[ \mathbf{n} \cdot \nabla r^2 = \sum_i \frac{\partial}{\partial x_i} (r^2) = \sum_i 2x_i \frac{\partial r}{\partial x_i} = 2r \frac{\partial r}{\partial x_i} \]

So,

\[ \int_{\mathcal{S}} \mathbf{n} \cdot \nabla r^2 \, dS = \int_{\mathcal{V}} \nabla \cdot (\nabla r^2) \, dV \]

Since \( \frac{\partial r}{\partial x_i} = x_i/r \) (Homework 5),

\[ = \int_{\mathcal{V}} 2 \mathbf{n} \cdot \mathbf{x} \, dV = 2 \int_{\mathcal{V}} \frac{\partial x_i}{\partial x_i} \, dV \text{ by the Divergence Theorem} \]

\[ = 2 \cdot \delta_{ii} \int_{\mathcal{V}} dV = V = \text{volume of domain} \]

\[ \text{where } V = \text{volume of domain bounded by } \mathcal{S}. \]
e. For those of you who would like more exercises, show

(i) \[ \int_{S} n \cdot \nabla (r^2) \, dS = 0 \]

(ii) \[ \int_{S} n \cdot \nabla (x \times a) \, dS = 0 \] where \( a \) is a constant vector and \( x \) is a position vector to a pt on the surface \( r = |x| \)

You will not be held responsible for this.

6. There is one final important point concerning the Divergence Theorem and that is if there are multiple bounding surfaces, you must include all of them.

In other words \[ \int_{V} \nabla \cdot f \, dV = \int_{\partial V} n \cdot f \, dS \]

So, if \( V \) is as shown below:

[Diagram showing multiple bounding surfaces]

Then \[ \int_{S} n \cdot f \, dS = \int_{S_1} n \cdot f \, dS + \int_{S_2} n \cdot f \, dS + \int_{S_3} n \cdot f \, dS + \int_{S_4} n \cdot f \, dS \]

and notice how on each surface, \( n \) points outward from \( V \)
7. **Stokes' Theorem** - This allows one to express an integral around a closed curve as an integral over that area minus the curve as a boundary.

Let \( C \) be a closed curve and let \( S \) be a surface with \( C \) as a bounding edge. Imagine a "hat-shaped" surface, let \( \mathbf{n} \) be unit normal to \( S \) with direction given by the right-hand rule – as you curl your hand in the direction indicated about \( C \), your thumb points in the direction of \( \mathbf{n} \).

\[ \mathbf{t} = \text{unit tangent vector to } C. \]

**Stokes' Theorem:**

\[
\oint_C f \cdot \mathbf{t} \, ds = \int_S \nabla \times \mathbf{F} \cdot d\mathbf{S}
\]

Recall proof given in Math 21.

Using vector notation:

\[ \oint_C f \cdot \mathbf{t} \, ds = \int_S \left[ \sum_{i=1}^{n} \frac{\partial f_i}{\partial x_i} \right] \, dS \]

Writing this all out:

\[
\oint_C (f_1 \mathbf{t}_1 + f_2 \mathbf{t}_2 + f_3 \mathbf{t}_3) \, ds = \int_S \left[ \frac{\partial}{\partial x_2} f_2 - \frac{\partial}{\partial x_3} f_3 - \frac{\partial}{\partial x_1} f_1 \right] \, dS
\]

**Note:** This is true for an arbitrary surface \( S \) with \( C \) as a bounding edge.

b. We can actually construct a simple proof using results developed so far.

Using \( s \) to denote arclength along the bounding curve \( C \),

\[ \frac{ds}{dx} = 1 \quad \text{then} \quad dx = t_i \, ds \]

So, considering the \( f_1 \) term:

\[ \oint_C f_1 \, ds = \oint_C f_1 \, dx_1 = \oint_{C'} f_1(x_1(x_2, x_3), x_2(x_3)) \, dx_1 \]

Function of \( x_1, x_2, x_3 \) represents curve \( C \) given \( C' \) in xy-plane and we have written

\[ f_1(x_1, x_2, x_3(x_1, x_2)) = f_1(x_1, x_2) \]

where we have essentially projected information down to the xy-plane.
b. "simple proof" (continued)

With information "in the plane" we can now use the identity given on pg. 68:

one form of the 'Divergence Theorem' was:

\[ \int_S \rho \cdot \Phi \; dS = \int_V \frac{\partial \Phi}{\partial x_i} \; dV \]

which has the 'planar version'

\[ \int_A \frac{\partial \Phi}{\partial x_i} \; dA = \int_C \Phi \cdot n_i \; ds \]

or

\[ \int_A \frac{\partial \Phi}{\partial x_1} \; dA = \int_C \Phi \; ds \quad \text{and} \quad \int_A \frac{\partial \Phi}{\partial x_2} \; dA = -\int_C \Phi \; dx_1 \]

\[ \int_A \frac{\partial \Phi}{\partial x_1} \; dA = \int_C \Phi \; ds \quad \text{and} \quad \int_A \frac{\partial \Phi}{\partial x_2} \; dA = -\int_C \Phi \; dx_1 \]

hence, beginning with the eqn on the bottom of the last page,

\[ \oint_C \Phi \; dx_1 = -\int_A \frac{\partial \Phi}{\partial x_2} \; dA \quad \text{but} \quad \frac{\partial}{\partial x_2} \left( x_1, x_2, x_3 \right) = \frac{\partial}{\partial x_2} \int_C \left( x_1, x_2, x_3 \left( x_1, x_2 \right) \right) \]

\[ = -\int_A \left[ \frac{\partial f_1}{\partial x_2} + \frac{\partial f_2}{\partial x_2} + \frac{\partial f_3}{\partial x_3} \right] \; dA \quad \text{by the chain-rule} \]

Now, a differential element \( dS \) in space is related to its projection in the \( xy \)-plane by \( dA = n_3 \; ds \) and furthermore one can show

\[ \frac{\partial x_3}{\partial x_2} = -\frac{n_2}{n_3} \]

so that we have

\[ \oint_C \Phi \; dx_1 = -\int_A \left[ \frac{\partial f_1}{\partial x_2} \; n_3 - \frac{\partial f_2}{\partial x_3} \; n_2 \right] \; dS \]

or

\[ \oint_C \Phi \; dx_1 = \oint_C \Phi \; t_1 \; ds = \int_S \left( \frac{\partial f_1}{\partial x_3} \; n_2 - \frac{\partial f_1}{\partial x_2} \; n_3 \right) \; dS \]

which accounts for two of the terms in Stokes' theorem. In a similar manner one can account for the other 2 terms. As in standard versions of these proofs, it is necessary to imagine that \( S \) is subdivided into sections which can be projected onto the \( xy \)-plane, even when the whole of \( S \) does not have a one-to-one projection.
8. Potentials for vector fields

a. Suppose that in some region \( \nabla \times \mathbf{f} = 0 \).

Then, from Stokes' theorem, it immediately follows that all line integrals between any two points \( P_1 \) and \( P_2 \) have the same value. In other words, \( \nabla \times \mathbf{f} = 0 \), so by Stokes' Theorem,

\[
\oint_C \mathbf{f} \cdot \mathbf{t} \, ds = 0
\]

and since

\[
\oint_C \mathbf{f} \cdot \mathbf{t} \, ds = \int_{C_1} \mathbf{f} \cdot \mathbf{t} \, ds - \int_{C_2} \mathbf{f} \cdot \mathbf{t} \, ds
\]

we've sketched the path \( C_2 \) as clockwise.

Clearly, such a vector field \( \mathbf{f} \) is called a CONSERVATIVE field.

b. Since this result is a SCALAR independent of path, we may define

a function \( \phi(x) \) such that

\[
\phi(x_2) - \phi(x_1) = \int_{x_1}^{x_2} \mathbf{f} \cdot dx
\]

provided that \( \nabla \times \mathbf{f} = 0 \).

Since the value of the integral is independent of path (i.e., only depends on the endpoints), then a differential change in \( \phi \) is given by

\[
d\phi = \mathbf{f} \cdot dx
\]

or since

\[
d\phi = \frac{\partial \phi}{\partial x_1} dx_1 + \frac{\partial \phi}{\partial x_2} dx_2 + \frac{\partial \phi}{\partial x_3} dx_3
\]

we identify

\[
\mathbf{f} = \nabla \phi
\]

\( \phi \) = potential function

(Sometimes a minus sign is inserted for convenience, i.e., people write \( \mathbf{f} = -\nabla \phi \).)

We also know that since \( \nabla \nabla \phi = 0 \) for any twice continuously differentiable function \( \phi \), then if \( \mathbf{f} = \nabla \phi \), we have \( \nabla \times \mathbf{f} = 0 \).

A vector function \( \mathbf{f} \) such that \( \nabla \times \mathbf{f} = 0 \) is called IRROTATIONAL.
4. Application: In the field of fluid dynamics, if
\[ \mathbf{v}(x) = \text{fluid velocity at } x, \]
the line integral
\[ \oint_C \mathbf{v} \cdot d\mathbf{x} \]
is called the circulation about \( C \).

Crudely, it provides a measure of the net rotation or 'circular motion' that occurs around some curve \( C \).

By Stokes' theorem
\[ \oint_C \mathbf{v} \cdot d\mathbf{x} = \iint_S \nabla \times \mathbf{v} \cdot d\mathbf{S} = \iint_S \mathbf{n} \cdot \omega \, dS \]
where \( \omega = \nabla \times \mathbf{v} \) is called the vorticity vector
and where \( S \) is any surface with \( C \) as a bounding edge.

Thus if the fluid is everywhere irrotational \( \omega = \nabla \times \mathbf{v} = 0 \),
then the circulation is zero for all curves \( C \).

Conversely, suppose it is known that the circulation is known to be zero for all curves \( C \). Then
\[ \iint_S \mathbf{n} \cdot (\nabla \times \mathbf{v}) \, dS = 0 \]
for all \( S \). Since this true for all possible \( S \), so that it must be true that \( \mathbf{n} \cdot (\nabla \times \mathbf{v}) = 0 \). But since the direction of \( \mathbf{n} \) is arbitrary given all possible surface, one has
\[ \nabla \times \mathbf{v} = 0 \quad \text{everywhere.} \]

Reading: Line & surface integrals = Hildebrand Sections 6.10-6.16
Greenberg Sections 9.1-9.6
9. Theoretical Applications of Integral Theorems

2. Derivation of the DIFFUSION EQN

Let's study a statement of conservation of energy for a heated material. We treat the material as continuous. In other words, we are concerned with length scales much larger than atomic distances.

Let $T(x, t) = \text{temperature at } x \text{ at time } t$

$q = \text{"heat flux vector" } [\text{has units: energy/area \cdot time}]$

$\rightarrow \text{Fourier's law of heat conduction: } q = -K \nabla T$

$k = \text{thermal conductivity } (k > 0)$

(heat flows from high $\rightarrow$ low temperature)

Let $\rho = \text{density (mass/volume)}$

$C_p = \text{heat capacity per unit mass (energy/cal\text{\textdegree})}$

$\text{Now consider an arbitrary fixed volume in the body.}$

The principle of conservation of energy states that (no sources of energy)

\[
\left\{ \text{rate of energy input into } V \right\}_{\text{across } S} - \left\{ \text{rate of energy output from } V \right\}_{\text{across } S} = \left\{ \text{time-rate-of-change of the total energy} \right\}_{\text{in } V}
\]

or

\[
\int \rho C_p \frac{\partial T}{\partial t} \, dV = \int_{S} (-q \cdot n) \, dS
\]

or

\[
\int (\rho C_p \frac{\partial T}{\partial t} + \nabla \cdot q) \, dV = 0
\]

This must be true for any arbitrary volume element $V$ in the body. This requires that the integrand be point-wise zero; i.e.,

\[
\rho C_p \frac{\partial T}{\partial t} + \nabla \cdot q = 0 \quad \rightarrow \quad \rho C_p \frac{\partial T}{\partial t} = -\nabla \cdot q
\]

or using Fourier's law, $q = -k \nabla T$, (and assuming $k$ = constant, independent of temperature)

\[
\rho C_p \frac{\partial T}{\partial t} = k \nabla^2 T
\]

\[
\frac{\partial T}{\partial t} = \frac{k}{\rho C_p} \nabla^2 T
\]

HEAT or DIFFUSION EQN
Derivation of Maxwell's eqn using Stokes Theorem

Recall from physics that Faraday discovered that electric fields can be induced by changing magnetic fields.

It is often stated in words something like

\[
\begin{align*}
\text{time-rate-of-change of the magnetic flux across a surface } S &= \text{electromotive force (emf)} \\
\text{across the circuit surrounding the surface } S \text{ recall that the emf is basically the total tangential force exerted on a charge around the loop or the tangential force per unit change (}= E \cdot t)\text{ integrated around the circuit}
\end{align*}
\]

\[
E = \text{electric field} \quad B = \text{magnetic field}
\]

\[
E(t, \xi) \quad B(t, \xi)
\]

or, mathematically,

\[
\frac{d}{dt} \int \mathbf{n} \cdot \mathbf{B} \, dS = -\oint \mathbf{E} \cdot d\mathbf{t} \quad \text{over circuit } C
\]

or by Stokes Theorem

\[
-\int \mathbf{n} \cdot (\nabla \times \mathbf{E}) \, dS
\]

If we assume \(S\) to be an arbitrary fixed surface then because it is fixed the order of differentiation and integration may be interchanged so that we have

\[
\int \mathbf{n} \cdot \left( \frac{\partial \mathbf{B}}{\partial t} + \nabla \times \mathbf{E} \right) \, dS = 0
\]

and this must be true for all surfaces \(S\) and all possible \(n\).

Hence, one concludes

\[
\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}
\]

Also, the electric field in a medium depends on the charge distribution according to

\[
\nabla \cdot \mathbf{E} = \rho / \varepsilon_0 \quad \text{where } \rho = \text{charge density} \quad (\text{charge/volume})
\]

\[
\varepsilon_0 = \text{permittivity of free space.}
\]
E. Vector Calculus Using Orthogonal Curvilinear Coordinates

1. So far we have only discussed the description of vectors and vector calculus relative to a cartesian coordinate system. We now wish to extend these ideas to other orthogonal coordinate systems. The two most common situations to keep in mind are:

   \[ \begin{align*}
   x_1 &= r \cos \theta \\
   x_2 &= r \sin \theta \\
   x_3 &= \phi
   \end{align*} \]

   **Cylindrical coordinates**

   \[ \begin{align*}
   x_1 &= r \sin \phi \cos \theta \\
   x_2 &= r \sin \phi \sin \theta \\
   x_3 &= r \cos \phi
   \end{align*} \]

   **Spherical coordinates**

   Note: Some texts interchange the role of \( \theta, \phi \).

   We see that in each of these coordinate systems, the unit base vectors \( (\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_\phi) \) or \( (\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_\phi) \) vary with position in space, i.e., they are functions of spatial position and this will mean that some care will be necessary when we speak of differentiation in these coordinate systems.

2. What do we really mean by a base vector?

   The unit base vector is in the direction associated with a small increment in the coordinate.

   For example, consider cylindrical coordinates:

   \[ \mathbf{\hat{r}} = \cos \theta \mathbf{e}_1 + \sin \theta \mathbf{e}_2 \]

   where

   \[ \frac{\partial \mathbf{r}}{\partial \theta} = \frac{\partial \mathbf{r}}{\partial \theta} \]

   Then

   \[ \mathbf{e}_r = \frac{\partial \mathbf{r}}{\partial \theta} = \frac{\cos \theta \mathbf{e}_1 + \sin \theta \mathbf{e}_2}{\sqrt{\cos^2 \theta + \sin^2 \theta}} \]

   Exercise: Find \( \mathbf{e}_\theta, \mathbf{e}_\phi \) in spherical coordinates.

   Aba clearly,

   \[ \mathbf{e}_\theta = \mathbf{e}_3 \]

   **NOTICE:** The base vectors are functions of position but are mutually orthogonal. Such coordinate systems are called **ORTHOGONAL CURVILINEAR COORDINATES.**
3. Now, it is possible to give a very careful and complete discussion of vector operations for a general, orthogonal, curvilinear, coordinate system.

In order to at least see clearly what the idea is, we first consider in detail vector operations in cylindrical coordinates.

\[ \vec{\nabla} \text{ in cylindrical coordinates} \]

Perhaps the first way one would think to proceed is to make a change of variables from \((x, y, z)\) to \((r, \theta, z)\). We will do this now, but it is somewhat longwinded.

\[ \begin{align*}
\vec{e}_r &= \cos \theta \, \vec{e}_x + \sin \theta \, \vec{e}_y \\
\vec{e}_\theta &= -\sin \theta \, \vec{e}_x + \cos \theta \, \vec{e}_y \\
\vec{e}_z &= \frac{\partial}{\partial z}
\end{align*} \] (1)

From the previous page, \( \vec{e}_r = \cos \theta \, \vec{e}_x + \sin \theta \, \vec{e}_y \); \( \vec{e}_\theta = -\sin \theta \, \vec{e}_x + \cos \theta \, \vec{e}_y \).

Let us see:

\[ \begin{align*}
\vec{e}_x &= \cos \theta \, \vec{e}_r - \sin \theta \, \vec{e}_\theta \\
\vec{e}_y &= \sin \theta \, \vec{e}_r + \cos \theta \, \vec{e}_\theta
\end{align*} \] (2)

Then, by the chain rule, \((x = r \cos \theta, y = r \sin \theta)\)

\[ \begin{align*}
\frac{\partial}{\partial x} &= \frac{\partial}{\partial r} \cos \theta + \frac{\partial}{\partial \theta} \frac{\partial}{\partial \theta} \\
\frac{\partial}{\partial y} &= \frac{\partial}{\partial r} \sin \theta + \frac{\partial}{\partial \theta} \frac{\partial}{\partial \theta}
\end{align*} \] (3)

where we have used \( r = \sqrt{x^2 + y^2} \), \( \theta = \tan^{-1} \frac{y}{x} \) to find:

\[ \begin{align*}
\frac{\partial r}{\partial x} &= \frac{x}{r^2} = \cos \theta \\
\frac{\partial r}{\partial y} &= \frac{y}{r^2} = \sin \theta
\end{align*} \]

\[ \begin{align*}
\frac{\partial \theta}{\partial x} &= -\frac{y}{r^2} = -\frac{\sin \theta}{r} \\
\frac{\partial \theta}{\partial y} &= \frac{x}{r^2} = \frac{\cos \theta}{r}
\end{align*} \]

Substituting (2) & (3) into (1) yields

\[ \vec{\nabla} = \vec{e}_r \frac{\partial}{\partial r} + \vec{e}_\theta \frac{1}{r} \frac{\partial}{\partial \theta} + \vec{e}_z \frac{\partial}{\partial z} \]

**NOTE:** It is very common for us to forget the \( \frac{1}{r} \) in front of \( \frac{\partial}{\partial \theta} \). Because \( \vec{\nabla} \) has units \( 1/\text{length} \), each term must also have dimensions of \( 1/\text{length} \).
2. Now, you might wish to bypass some algebra so an alternative is to first note that, we expect

\[ \nabla = \varepsilon_r \frac{\partial}{\partial r} + \varepsilon_\theta \frac{\partial}{\partial \theta} + \varepsilon_z \frac{\partial}{\partial z} \]

and the functions \( h_r(r, \theta) \) and \( h_\theta(r, \theta) \).

Since \( \varepsilon_r, \varepsilon_\theta, \varepsilon_z \) are mutually orthogonal, then

\[ \varepsilon_r \cdot \nabla = h_r(r, \theta) \frac{\partial}{\partial r} \quad \text{and} \quad \varepsilon_\theta \cdot \nabla = h_\theta(r, \theta) \frac{\partial}{\partial \theta} \]

From pg. 88, \( \varepsilon_r \) is defined as \( \frac{2x}{\rho r} | \frac{2x}{\rho r} | \)

so that,

\[ h_r = \varepsilon_r \cdot \nabla = \frac{1}{\left| \frac{2x}{\rho r} \right|} \frac{2x}{\rho r} \cdot \nabla = \frac{2x}{\rho r} \quad \text{since} \quad \left| \frac{2x}{\rho r} \right| = 1 \quad (p. 89) \]

Also,

\[ h_\theta = \varepsilon_\theta \cdot \nabla = \frac{1}{\left| \frac{2x}{\rho \theta} \right|} \frac{2x}{\rho \theta} \cdot \nabla = \frac{1}{\left| \frac{2x}{\rho \theta} \right|} \frac{2x}{\rho \theta} = \frac{1}{r \theta} \]

from pg. 88

This was much simpler.

3. Important point: \( \nabla \) is now known in cylindrical coords

\[ \nabla = \varepsilon_r \frac{\partial}{\partial r} + \varepsilon_\theta \frac{1}{r} \frac{\partial}{\partial \theta} + \varepsilon_z \frac{\partial}{\partial z} \]

a. example: Suppose \( \Psi = \Psi(\theta, z) \) only, independent of \( r \).

Then

\[ \nabla \phi = \varepsilon_\theta \frac{1}{r} \frac{\partial \Psi}{\partial \theta} + \varepsilon_z \frac{\partial \Psi}{\partial z} \quad \text{and there is no} \quad r \text{-coordinate.} \]
4. **Divergence in cylindrical coordinates**

Now, let's calculate \( \nabla \cdot \mathbf{f} \).

a. In cylindrical coordinates, we write

\[
\mathbf{f}(r, \theta, z) = f_r(r, \theta, z) \mathbf{e}_r + \frac{1}{r} f_\theta(r, \theta, z) \mathbf{e}_\theta + f_z(r, \theta, z) \mathbf{e}_z
\]

Each of the components may be a function of position.

b. Formally, take \( \nabla \cdot \mathbf{f} \) and remember that in cylindrical coordinates, the unit vectors \( \mathbf{e}_r, \mathbf{e}_\theta \) are not constant vectors but vary with \( \theta \).

**Note:** \( \mathbf{e}_r(\theta); \mathbf{e}_\theta(\theta). \)

\[
\mathbf{e}_r(\theta + \Delta \theta) - \mathbf{e}_r(\theta) = \Delta \theta \mathbf{e}_\theta
\]

Similarly, geometrically show

\[
\frac{\partial \mathbf{e}_r}{\partial \theta} = \mathbf{e}_\theta, \quad \frac{\partial \mathbf{e}_\theta}{\partial \theta} = -\mathbf{e}_r
\]

Both these statements can be demonstrated directly from the equations on p. 88.

C. So, the calculation: proceed term-by-term—remember to take \( \frac{\partial}{\partial r} \) of the unit vector depending on \( r \).

\[
\nabla \cdot \mathbf{f} = \left( \frac{\partial}{\partial r} + \frac{\theta}{r} + \frac{\partial}{\partial \theta} \right) \left( f_r \mathbf{e}_r + \frac{1}{r} f_\theta \mathbf{e}_\theta + f_z \mathbf{e}_z \right)
\]

\[
= \mathbf{e}_r \cdot \frac{\partial}{\partial r} f_r + \frac{\mathbf{e}_r \cdot \frac{\partial}{\partial \theta} f_\theta}{r} + \frac{\mathbf{e}_r \cdot \frac{\partial}{\partial \theta} f_z}{r} + \ldots
\]

\[
\text{independent of } r
\]

\[
= \frac{\partial f_r}{\partial r} + \frac{\theta}{r} f_r + \frac{1}{r} \frac{\partial}{\partial \theta} \left( f_r \mathbf{e}_r + f_\theta \mathbf{e}_\theta \right) + \frac{\partial f_\theta}{\partial \theta}
\]

\[
\text{by inspection, these are}
\]

\[
\text{only terms that possible survive the inner product}
\]

\[
= \frac{\partial f_r}{\partial r} + \frac{\theta}{r} f_r + \frac{1}{r} \frac{\partial}{\partial \theta} f_r + 
\]

\[
\text{general result}
\]

\[
\nabla \cdot \mathbf{f} = \frac{\partial f_r}{\partial r} + \frac{f_r}{r} + \ldots \frac{\partial f_\theta}{\partial \theta} + \frac{\partial f_z}{\partial z}
\]

\[
\nabla \cdot \mathbf{f} = \frac{1}{r} \frac{\partial}{\partial r} (r f_r) + \frac{1}{r} \frac{\partial}{\partial \theta} + \frac{\partial}{\partial z}
\]
The discussion which follows on the next few pages is very general and closely follows Hildebrand. To try to understand the basic ideas involved, for example, try to think about this geometrically.

5. Let's proceed in a more general manner for any rectilinear coordinate system.

a. Let $\mathbf{e}_i$, $i=1, 2, 3$ be a general set of orthogonal coordinates.

Every point in space may be specified by giving the coordinates $(e_1, e_2, e_3)$. If we hold $e_1, e_2$ fixed and vary $e_3$, we sweep out the $e_3$-curve (similar statement for $e_1$-curves and $e_2$-curves).

We will only discuss the case where the coordinate lines (curves) are mutually orthogonal at each point in space.

Curvilinear coordinates.

b. Now introduce UNIT BASE VECTORS, $u_i$, $i=1, 2, 3$. (right-handed)

By definition, the base vector in the $i$-direction indicates the direction of change in the $\mathbf{e}_i$-coordinate, holding the other 2 coordinates fixed.

Let $\mathbf{x}(\mathbf{e}_i)$ denote the position vector associated with the $\mathbf{e}_i$-coordinate ($i=1, 2, 3$)

Then, by definition, $\frac{dx}{de_i}$ divide by length of $\frac{dx}{de_i}$, so $u_i = \frac{dx}{de_i}$

$u_i$ is of unit length.

Let $d\mathbf{s}$ = distance from $\mathbf{r}$ to $\mathbf{r} + d\mathbf{e}_i$ (or $\mathbf{r}'$ in figure above)

$d\mathbf{s}^2 = dx_i \cdot dx_i = \left(\frac{dx}{de_i} \right)^2 \cdot \frac{de_i}{de_i}$

We will require that the unit base vectors be ORTHOGONAL: $\frac{dx}{de_j} \cdot \frac{dx}{de_k} = 0$, $j \neq k$

$\Rightarrow d\mathbf{s}^2 = \frac{dx}{de_1}^2 \cdot d\mathbf{e}_1^2 + \frac{dx}{de_2}^2 \cdot d\mathbf{e}_2^2 + \frac{dx}{de_3}^2 \cdot d\mathbf{e}_3^2$

DEFINE: $h_i = \frac{dx}{de_i}$, $i=1, 2, 3$ <-> METRIC COEFFICIENTS or SCALE FACTORS

$\frac{dx}{de_i} = h_i u_i$ (no sum i) so $h_i$ provides a measure of distance along $\mathbf{e}_i$-curve

$d\mathbf{s}^2 = h_1^2 \cdot d\mathbf{e}_1^2 + h_2^2 \cdot d\mathbf{e}_2^2 + h_3^2 \cdot d\mathbf{e}_3^2 = h_i^2 \cdot d\mathbf{e}_i^2$ where now we've used summation convention but be careful.
C. As an example, consider cylindrical coordinates \((\xi_1 = r, \xi_2 = \theta, \xi_3 = z)\)

Position vector: \(\mathbf{r} = r \cos \theta \mathbf{e}_1 + r \sin \theta \mathbf{e}_2 + z \mathbf{e}_3\)

\(h_1 = h_2 = \left| \frac{\partial \mathbf{r}}{\partial \xi_1} \right| = \left| \cos \theta \mathbf{e}_1 + \sin \theta \mathbf{e}_2 \right| = 1\)

\(h_3 = \left| \frac{\partial \mathbf{r}}{\partial \xi_3} \right| = \left| r \sin \theta \mathbf{e}_1 + r \cos \theta \mathbf{e}_2 \right| = r\)

It follows that the square of a differential displacement \((d\mathbf{s})^2\), is given by

\[(d\mathbf{s})^2 = dr^2 + r^2 d\theta^2 + dz^2\]

d. Volume elements

A small element of volume is formed by small displacements along the \(\xi_1, \xi_2, \) and \(\xi_3\) curves. Recall that the metric coefficients \(h_i\) provide a measure of length or distance along the \(\xi_i\) curve.

\(dV\) is a differential element of volume. It is the parallelepiped formed by the three orthogonal vector displacements

\[
\left( \begin{array}{c}
\frac{\partial s_1}{\partial \xi_1} \\
\frac{\partial s_2}{\partial \xi_2} \\
\frac{\partial s_3}{\partial \xi_3}
\end{array} \right)
\]

or

\[
\left( \begin{array}{c}
h_1 d\xi_1 u_1 \\
h_2 d\xi_2 u_2 \\
h_3 d\xi_3 u_3
\end{array} \right)
\]

so that the element of volume follows from

\[
dV = \left( h_1 d\xi_1 u_1 \cdot h_2 d\xi_2 u_2 \cdot h_3 d\xi_3 u_3 \right) = h_1 h_2 h_3 d\xi_1 d\xi_2 d\xi_3
\]

e. Surface elements (similar to above discussion)

A differential surface element is described by differential displacements along two of the coordinate curves. \(h_2 d\xi_2 u_1\)

For example, if we consider the surface at which \(\xi_3 = \text{constant}\); \(\mathbf{u}_1 \perp \mathbf{s}\)

Then

\[
dS = \left| h_2 d\xi_2 u_1 \wedge h_3 d\xi_3 u_3 \right| = h_2 h_3 d\xi_2 d\xi_3
\]

Magnitude of cross product gives differential element of area in a plane tangent to the surface.
Example: Again use cylindrical coordinates

\[ dV = r \, dr \, d\theta \, dz \]

\[ dS = r \, dr \, d\theta \] on \( r = \) constant

\[ dS = \, dr \, dz \] on \( \theta = \) constant

\[ dS = r \, dr \, dz \] on \( r = \) constant

f. The representation of a vector - we can represent any vector by giving its components and unit vectors.

So,

\[ f = f_1 \mathbf{e}_1 + f_2 \mathbf{e}_2 + f_3 \mathbf{e}_3 \]

or

\[ f = f_1 \mathbf{u}_1 + f_2 \mathbf{u}_2 + f_3 \mathbf{u}_3 \]

NOTE: In general, \( f_i^{(u)} \neq f_i^{(e)} \)

and given one of these representation the components may be calculated via

\[ f_i^{(e)} = f_1 \mathbf{e}_1 \quad , \quad f_i^{(u)} = f_1 \mathbf{u}_1 \]

It is important to keep in mind that the vector \( f \) is the same. It is only the representations of the vector (or, equivalently, its description via components and unit vectors) which differ.

6. Gradient of a scalar in curvilinear coordinates

Let us define \( \nabla \phi \) via

\[ d\phi = \nabla \phi \cdot d\mathbf{x} \]

differential vector displacement

\[ \nabla \phi = \lambda_1 \mathbf{u}_1 + \lambda_2 \mathbf{u}_2 + \lambda_3 \mathbf{u}_3 \quad \text{\( \Rightarrow \) Now, what are the \( \lambda \)'s?} \]

Well,

\[ dx = \frac{dx}{d\xi_1} \, d\xi_1 + \frac{dx}{d\xi_2} \, d\xi_2 + \frac{dx}{d\xi_3} \, d\xi_3 = h_1 \, d\xi_1 \, u_1 + h_2 \, d\xi_2 \, u_2 + h_3 \, d\xi_3 \, u_3 \]

So,

\[ d\phi = \nabla \phi \cdot dx = \lambda_1 \, h_1 \, d\xi_1 + \lambda_2 \, h_2 \, d\xi_2 + \lambda_3 \, h_3 \, d\xi_3 \]

Hence:

\[ \lambda_1 = \frac{1}{h_1} \frac{d\phi}{d\xi_1} \quad \lambda_2 = \frac{1}{h_2} \frac{d\phi}{d\xi_2} \quad \lambda_3 = \frac{1}{h_3} \frac{d\phi}{d\xi_3} \]

Therefore, in general, the gradient may be represented as

\[ \nabla \phi = u_1 \frac{1}{h_1} \frac{d\phi}{d\xi_1} + u_2 \frac{1}{h_2} \frac{d\phi}{d\xi_2} + u_3 \frac{1}{h_3} \frac{d\phi}{d\xi_3} \]

(1)
Example 1: Gradient of a scalar function in cylindrical coordinates

\[ \nabla \phi = \hat{e}_r \frac{\partial \phi}{\partial r} + \hat{e}_\theta \frac{\partial \phi}{\partial \theta} + \hat{e}_z \frac{\partial \phi}{\partial z} \]

It is very useful to notice that \( \nabla \phi \) represents a vector function whose components represent derivatives with respect to distance in a certain direction. Hence, in each component, we must basically be dividing by a length. In the above, the \( r, \theta \) components clearly represent derivatives with respect to distance and in the \( \theta \)-direction the distance is \( \text{"r \, d\theta"} \).

In other words, it is useful to think of \( \nabla \phi \) as

\[ \nabla \phi = u_1 \frac{\partial \phi}{\partial s_1} + u_2 \frac{\partial \phi}{\partial s_2} + u_3 \frac{\partial \phi}{\partial s_3} \]

where \( ds_1 = h_1 \, ds \), is the distance along the \( e_1 \) axis for \( ds_2 \), \( ds_3 \) (\( ds_2 = ds_3 = 0 \)) and similarly for \( ds_2 \) and \( ds_3 \).

7. Divergence of a Vector Function \( \nabla \chi \) in curvilinear coordinates

a. As a preliminary, we notice the following:

Take \( \phi = \xi_i \) in eqn (1) on pg 94, where \( \xi_i \) is any one of the coordinates, then

\[ \nabla \xi_i = \frac{\partial \xi_i}{\partial s} \]

for the \( i \)th coordinate (DO NOT USE THE SUMMATION CONVENTION HERE)

Next, for this right-handed coordinate system,

\[ u_1 = u_2 \times u_3 \implies \frac{u_1}{h_2 h_3} = \frac{u_2}{h_2} \times \frac{u_3}{h_3} = \nabla \xi_2 \times \nabla \xi_3 \]

IDENTITY: \( \nabla \times (\nabla \xi_i \times \nabla \xi_j) = 0 \) for any twice continuously differentiable functions \( \xi_1, \xi_2 \)

Exercise: prove this identity.

So, clearly

\[ \nabla \cdot (\nabla \xi_2 \times \nabla \xi_3) = 0 \quad \text{or} \quad \nabla \cdot \left( \frac{u_1}{h_2 h_3} \right) = 0 \]

Likewise,

\[ \nabla \cdot \left( \frac{u_2}{h_1 h_3} \right) = 0 \quad \nabla \cdot \left( \frac{u_3}{h_1 h_2} \right) = 0. \]
b. Now consider $\nabla \cdot \mathbf{f}$.

In general, \[ \mathbf{f} = f_1 \mathbf{u}_1 + f_2 \mathbf{u}_2 + f_3 \mathbf{u}_3 \]
so, we can write

\[ \nabla \cdot \mathbf{f} = \nabla \cdot (f_1 \mathbf{u}_1) + \nabla \cdot (f_2 \mathbf{u}_2) + \nabla \cdot (f_3 \mathbf{u}_3) \]

\[ = \nabla \cdot \left( \frac{h_2 h_3 f_1}{h_2 h_3} \mathbf{u}_1 \right) + \nabla \cdot \left( \frac{h_1 h_3 f_2}{h_1 h_3} \mathbf{u}_2 \right) + \nabla \cdot \left( \frac{h_1 h_2 f_3}{h_1 h_2} \mathbf{u}_3 \right) \]

But, \[ \nabla = \frac{\partial}{\partial x_1} \mathbf{u}_1 + \frac{\partial}{\partial x_2} \mathbf{u}_2 + \frac{\partial}{\partial x_3} \mathbf{u}_3 \]
so that, for example,

\[ \nabla \cdot \left( \frac{h_2 h_3 f_1}{h_2 h_3} \mathbf{u}_1 \right) = \frac{\partial}{\partial x_1} \left( \frac{h_2 h_3 f_1}{h_2 h_3} \right) = \frac{1}{h_1 h_2 h_3} \frac{\partial}{\partial x_1} \left( h_2 h_3 f_1 \right) \]

\[ = \frac{1}{h_1 h_2 h_3} \frac{\partial}{\partial x_1} \left( h_2 h_3 f_1 \right) \]

Similarly for the other terms

\[ \nabla \cdot \mathbf{f} = \frac{1}{h_1 h_2 h_3} \left[ \frac{\partial}{\partial x_1} \left( h_2 h_3 f_1 \right) + \frac{\partial}{\partial x_2} \left( h_1 h_3 f_2 \right) + \frac{\partial}{\partial x_3} \left( h_1 h_2 f_3 \right) \right] \quad (1) \]

Example: Cylindrical coordinates \( h_1 = h_r = 1 \), \( h_2 = h_\theta = r \), \( h_3 = h_z = 1 \)

\[ \mathbf{f} = f_r \mathbf{e}_r + f_\theta \mathbf{e}_\theta + f_z \mathbf{e}_z \Rightarrow \nabla \cdot \mathbf{f} = \frac{1}{r} \left[ \frac{\partial}{\partial r} (r f_r) + \frac{\partial}{\partial \theta} (r f_\theta) + \frac{\partial}{\partial z} (r f_z) \right] \]

Hence,

\[ \nabla \cdot \mathbf{f} = \frac{1}{r} \frac{\partial}{\partial r} (r f_r) + \frac{1}{r} \frac{\partial}{\partial \theta} (r f_\theta) + \frac{1}{r} \frac{\partial}{\partial z} (r f_z) \]

Example: Spherical coordinates \((r, \phi, \theta)\)

\[ h_1 = h_r = 1 \]
\[ h_2 = h_\phi = r \]
\[ h_3 = h_\theta = r \sin \phi \]

\[ \nabla \cdot \mathbf{f} = \frac{1}{r^2 \sin \phi} \left[ \frac{\partial}{\partial r} \left( r^2 \sin \phi f_r \right) + \frac{\partial}{\partial \theta} \left( r \sin \phi f_\phi \right) + \frac{\partial}{\partial \phi} \left( r f_\theta \right) \right] \]

Or

\[ \nabla \cdot \mathbf{f} = \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 f_r) + \frac{1}{r} \frac{\partial}{\partial \phi} (r \sin \phi f_\phi) + \frac{1}{r} \frac{\partial}{\partial \theta} (r f_\theta) \]
8. The Laplacian  \( \nabla^2 \phi = \nabla \cdot \nabla \phi \)

Well we know how to take \( \nabla \cdot f \), so let \( f = \nabla \phi \)

\[ f = \nabla \phi = u_1 \frac{\partial \phi}{\partial x_1} + u_2 \frac{\partial \phi}{\partial x_2} + u_3 \frac{\partial \phi}{\partial x_3} = f_1 u_1 + f_2 u_2 + f_3 u_3 \]

Hence, substituting into eqn (4) on p. 82,

\[
\nabla^2 \phi = \frac{1}{h_1 h_2 h_3} \left[ \frac{\partial}{\partial x_1} \left( \frac{h_2 h_3}{h_1} \frac{\partial \phi}{\partial x_1} \right) + \frac{\partial}{\partial x_2} \left( \frac{h_1 h_3}{h_2} \frac{\partial \phi}{\partial x_2} \right) + \frac{\partial}{\partial x_3} \left( \frac{h_1 h_2}{h_3} \frac{\partial \phi}{\partial x_3} \right) \right]
\]

Example: cylindrical coordinates (\( r, \theta, z \)): \( h_1 = h_2 = 1 \), \( h_3 = r \), \( \phi = \phi \)

\[
\nabla^2 \phi = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial \phi}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 \phi}{\partial \theta^2} + \frac{\partial^2 \phi}{\partial z^2}
\]

9. The Curl \( \nabla \times f \)

a. Once again, we begin with a useful preliminary

We know that \( \nabla \times \nabla \phi = 0 \) for any scalar function \( \phi \) so certainly

\[ \nabla \times \nabla F_i = 0 \quad \text{for} \quad i = 1, 2, 3 \quad (F_i \text{ is one of the coordinates}) \]

But

\[ \nabla F_i = \frac{u_i}{h_i} \quad \text{for} \quad i = 1, 2, 3 \quad \text{(do not sum)} \rightarrow \text{see p. 81} \]

so that

\[ \nabla \times \left( \frac{u_i}{h_i} \right) = 0 \quad \text{for} \quad i = 1, 2, 3 \]

b. It is now straightforward to calculate \( \nabla \times f \) as shown below:

\[
f = f_1 u_1 + f_2 u_2 + f_3 u_3
\]

For example, first consider \( \nabla \times (f_1 u_1) \)

\[
\nabla \times (f_1 u_1) = \nabla \times \left( h_1 f_1 \frac{u_1}{h_1} \right) = \nabla (h_1 f_1) \times \frac{u_1}{h_1} + h_1 f_1 \nabla \times \left( \frac{u_1}{h_1} \right)
\]

(we have made use of the fact that \( \nabla \times (\phi q) = \nabla \phi \times q + \phi \nabla \times q \))

-you should be able to prove this
9. The curl (continued)

Making use of the general definition of $\nabla \cdot (p \cdot \Phi)$

$\nabla \times (f_1 \hat{u}_1) = \nabla \times (h_1 f_1) = \frac{u_1}{h_1} \left[ \frac{\partial}{\partial x_3} (h_1 f_1) + \frac{u_2}{h_2} \frac{\partial}{\partial x_2} (h_1 f_1) + \frac{u_3}{h_3} \frac{\partial}{\partial x_1} (h_1 f_1) \right] \frac{u_1}{h_1}$

$\Rightarrow \quad \nabla \times (f_1 \hat{u}_1) = u_1 \frac{1}{h_1 h_3} \frac{\partial}{\partial x_3} (h_1 f_1) - u_2 \frac{1}{h_1 h_2} \frac{\partial}{\partial x_2} (h_1 f_1) \quad \text{since} \quad u_2 \hat{u}_2 = u_3 \hat{u}_3 \quad u_3 \hat{u}_3 = u_2 \hat{u}_2$

The other two terms of $\nabla \times f$ are calculated in a similar manner.

The general result is:

$\nabla \times f = \frac{1}{h_2 h_3} \left[ \frac{\partial}{\partial x_2} (h_3 f_3) \frac{\partial}{\partial x_3} (h_2 f_2) \right] + \frac{u_2}{h_1 h_3} \left[ \frac{\partial}{\partial x_3} (h_1 f_1) - \frac{\partial}{\partial x_1} (h_3 f_3) \right]$

$+ \frac{u_3}{h_1 h_2} \left[ \frac{\partial}{\partial x_1} (h_2 f_2) - \frac{\partial}{\partial x_2} (h_1 f_1) \right]$

which can be expressed as

$\nabla \times f = \frac{1}{h_2 h_3 h_4} \det \begin{vmatrix} h_1 & u_1 & h_2 & u_2 & h_3 & u_3 \\ \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} & \frac{\partial}{\partial x_4} \\ h_1 & h_2 & h_3 & h_4 \\ f_1 & f_2 & f_3 & f_4 \end{vmatrix}$

$\Rightarrow$ The special case of cylindrical coordinates ($r, \theta, z$) is straightforward:

$h_1 = h_r = 1, \quad h_2 = h_\theta = r, \quad h_3 = h_1, \quad f_1 = f_r, \quad f_2 = f_\theta, \quad f_3 = f_z$

$\nabla \times f = r \left[ \frac{1}{r} \frac{\partial}{\partial r} - \frac{\partial}{\partial \theta} \right] + z \left[ \frac{\partial}{\partial \theta} - \frac{\partial}{\partial z} \right] + \frac{1}{r} \left[ \frac{\partial}{\partial r} - \frac{\partial}{\partial \theta} \right]$

Similarly, in spherical coordinates ($h_1 = h_r = 1, \quad h_2 = h_\theta = r, \quad h_3 = h_\phi = r s, \quad f_1 = f_r, \quad f_2 = f_\theta, \quad f_3 = f_\phi$)

$\nabla \times f = \frac{1}{r \sin \theta} \left[ \frac{\partial}{\partial \theta} (\sin \theta f_\theta) - \frac{\partial f_\phi}{\partial \theta} \right] + \frac{1}{rs \sin \phi} \left[ \frac{\partial}{\partial \phi} - \sin \phi \frac{\partial}{\partial r} \right] (f_\phi)$

$+ \frac{1}{rs \sin \phi} \left[ \frac{\partial}{\partial r} (r f_\phi) - \frac{\partial f_r}{\partial \phi} \right]$
F. An Introduction to Tensors & Dyads.

1. Preliminary remarks

a. We now wish to generalize our ideas concerning vectors to objects called tensors. We will try both to describe some of the mathematics of tensors and show why and how they arise in physical situations.

b. The idea is we have previously described scalars & vectors as:

- scalar - characterized by magnitude
- vector - characterized by magnitude & direction

Now a 2nd order tensor - characterized by magnitude & two directions, (or a dyadic)

d. You have actually seen something very similar before.

For example, the vector \( \mathbf{a} \cdot \mathbf{c} \) could be written

\[
\mathbf{a} \mathbf{\cdot} \mathbf{c} \quad \text{(dyadic)}
\]

which has the property \( \mathbf{a} \mathbf{\cdot} \mathbf{c} = \mathbf{a} (\mathbf{b} \cdot \mathbf{c}) \)

This points out the following important property: using \( \mathbf{a} \mathbf{\cdot} \mathbf{c} \) notation, the quantity \( \mathbf{a} \mathbf{\cdot} \mathbf{c} \) may be written

\[
\mathbf{a} \mathbf{\cdot} \mathbf{c} = a_i \mathbf{e}_i \mathbf{\cdot} \mathbf{e}_j
\]

(\text{summation convention in use})

\[
\mathbf{a} \mathbf{\cdot} \mathbf{c} = a_i \delta_{ij}
\]

where the i\(j\) component 2 direction \( \delta_{ij} = 1,2,3 \)

These mathematical objects that require 2 directions (or 2 indices) to be defined often correspond to PHYSICAL situations where physical properties are different in different directions.

(see page 105)
2. Definition of a 2nd order dyadic

a. Define the 2nd order dyadic \( \mathbf{T} \) by \( \mathbf{T} = \mathbf{a} \mathbf{b} \)
and assign it the following operational properties:

(i) \( \mathbf{c} \cdot \mathbf{T} = (\mathbf{c} \cdot \mathbf{a}) \mathbf{b} \)

(ii) \( \mathbf{T} \cdot \mathbf{c} = \mathbf{a} (\mathbf{b} \cdot \mathbf{c}) \)

(iii) If \( \mathbf{T}, \mathbf{S}, \mathbf{R} \) are dyadics then we also define standard linear operations:

\[
\mathbf{c} \cdot (\mathbf{T} + \mathbf{S} + \mathbf{R} + \ldots) = \mathbf{c} \cdot \mathbf{T} + \mathbf{c} \cdot \mathbf{S} + \mathbf{c} \cdot \mathbf{R} + \ldots
\]

\[
(\mathbf{T} + \mathbf{S} + \mathbf{R} + \ldots) \cdot \mathbf{c} = \mathbf{T} \cdot \mathbf{c} + \mathbf{S} \cdot \mathbf{c} + \mathbf{R} \cdot \mathbf{c} + \ldots
\]

(as you would expect)

NOTE: a vector can be thought of as a first order dyadic or tensor
and a scalar as a 0th order tensor.

⇒ The inner product of dyadic and a vector produces a vector.

⇒ ORDER IS IMPORTANT: \( \mathbf{c} \cdot \mathbf{T} \neq \mathbf{T} \cdot \mathbf{c} \)

b. Using index notation we write

\( \mathbf{a} = a^i \mathbf{e}_i \)
\( \mathbf{b} = b^j \mathbf{e}_j \)

\[
\mathbf{T} = a^i b^j \mathbf{e}_i \mathbf{e}_j
\]

So:

\( \mathbf{c} \cdot \mathbf{T} = c^k a^i b^j \mathbf{e}_i \mathbf{e}_j \mathbf{e}_k \)

\( \mathbf{T} \cdot \mathbf{c} = a^i b^j \mathbf{e}_i \mathbf{e}_j c^k \mathbf{e}_k \)

Alternatively, we can discuss

\( \mathbf{T} = T^i j \mathbf{e}_i \mathbf{e}_j \)

and speak of the 2nd order tensor \( \mathbf{T} \).


NOTATION: I denote a 2nd order tensor using 2 underlines
(generally, vectors will be lower case and 2nd order tensors uppercase)

and require 2 indices to characterise completely


100
a simple idea of a (linear operator) : The work done by a force acting through a displacement \( d \) is \( f \cdot d \)
and so you could choose to think about \( f \) as a linear operator
that, once fed the displacement \( d \), yields a scalar we call work, \( f \cdot d \).

3. 2nd order tensors : An alternative way of thinking about things

a. A 2nd order tensor can be thought of as a "machine"
that has a vector for its input and outputs another vector:

\[
\begin{align*}
\begin{array}{c}
\text{THE TENSOR} \\
\text{MACHINE}
\end{array}
\rightarrow \begin{array}{c}
\text{b which we denote}
\end{array}
\end{align*}
\]

\[
\begin{align*}
\mathbf{T} & : \mathbf{a} \rightarrow \mathbf{b} \\
\text{\( T \) operating on a vector produces another vector.}
\end{align*}
\]

b. Cartesian components of a 2nd rank tensor

Let's consider an arbitrary vector \( \mathbf{c} = c_x \mathbf{e}_x + c_y \mathbf{e}_y + c_z \mathbf{e}_z \)

Then, from a formal (operational) viewpoint,

\[
\begin{align*}
\mathbf{T} \cdot \mathbf{c} &= c_x \mathbf{T} \cdot \mathbf{e}_x + c_y \mathbf{T} \cdot \mathbf{e}_y + c_z \mathbf{T} \cdot \mathbf{e}_z \\
&= \begin{bmatrix}
T_{xx} & T_{xy} & T_{xz} \\
T_{yx} & T_{yy} & T_{yz} \\
T_{zx} & T_{zy} & T_{zz}
\end{bmatrix}
\begin{bmatrix}
c_x \\
c_y \\
c_z
\end{bmatrix}
\end{align*}
\]

So, for example, we choose to write

\[
\mathbf{T} \cdot \mathbf{e}_x = T_{xx} \mathbf{e}_x + T_{yx} \mathbf{e}_y + T_{zx} \mathbf{e}_z \quad \text{which is a vector.}
\]

and similarly

\[
\begin{align*}
\mathbf{T} \cdot \mathbf{e}_y &= T_{xy} \mathbf{e}_x + T_{yy} \mathbf{e}_y + T_{zy} \mathbf{e}_z \\
\mathbf{T} \cdot \mathbf{e}_z &= T_{xz} \mathbf{e}_x + T_{yz} \mathbf{e}_y + T_{zz} \mathbf{e}_z
\end{align*}
\]

Thus, since \( c_x = \mathbf{c} \cdot \mathbf{e}_x \), \( c_y = \mathbf{c} \cdot \mathbf{e}_y \), \( c_z = \mathbf{c} \cdot \mathbf{e}_z \)

we can write eqn (i) as

\[
\begin{align*}
\mathbf{T} \cdot \mathbf{c} &= T_{xx} c_x \mathbf{e}_x + T_{xy} c_y \mathbf{e}_y + T_{xz} c_z \mathbf{e}_z \\
&+ T_{yx} c_x \mathbf{e}_x + T_{yy} c_y \mathbf{e}_y + T_{zy} c_z \mathbf{e}_z \\
&+ T_{zx} c_x \mathbf{e}_x + T_{zy} c_y \mathbf{e}_y + T_{zz} c_z \mathbf{e}_z \\
&\quad \text{slight rearrangement}
\end{align*}
\]

\[
\begin{align*}
\mathbf{T} \cdot \mathbf{c} &= \begin{bmatrix}
T_{xx} & T_{xy} & T_{xz} \\
T_{yx} & T_{yy} & T_{yz} \\
T_{zx} & T_{zy} & T_{zz}
\end{bmatrix}
\begin{bmatrix}
c_x \\
c_y \\
c_z
\end{bmatrix} \\
&= \begin{bmatrix}
T_{xx} c_x + T_{xy} c_y + T_{xz} c_z \\
T_{yx} c_x + T_{yy} c_y + T_{yz} c_z \\
T_{zx} c_x + T_{zy} c_y + T_{zz} c_z
\end{bmatrix} \cdot \mathbf{c}
\end{align*}
\]

The set of 9 quantities \( T_{ij} \) are called the Cartesian components of the 2nd order tensor \( T \).

Cartesian representation of the 2nd order tensor \( T \)

\[
\begin{bmatrix}
\mathbf{e}_x \mathbf{e}_x & \mathbf{e}_x \mathbf{e}_y & \mathbf{e}_x \mathbf{e}_z \\
\mathbf{e}_y \mathbf{e}_x & \mathbf{e}_y \mathbf{e}_y & \mathbf{e}_y \mathbf{e}_z \\
\mathbf{e}_z \mathbf{e}_x & \mathbf{e}_z \mathbf{e}_y & \mathbf{e}_z \mathbf{e}_z
\end{bmatrix}
\]

is a basis for the 2nd order tensors.
3. 2nd order tensors (continued)

c. Some sample representations using index notation

\[ c \cdot T = c_i e_i \cdot T_{jk} e_j e_k = c_i T_{ik} e_k \quad (a \ vector) \]

\[ T \cdot c = T_{ij} e_i e_j \cdot c_k e_k = T_{ij} c_j e_i \quad (a \ vector) \]

→ RULES: Nesting Convention ⇒ vector operations occur between the closest pair of unit vectors

Example:

\[ T^\wedge e = T_{ij} e_i e_j \wedge c_k e_k = T_{ij} c_k e_i (e_j \wedge e_k) = T_{ij} c_k e_{jk} e_i e_m \]

⇒ order of unit vectors is now important.

d. Notice the similarity with matrices: A 3x3 matrix

\[
\begin{bmatrix}
1 & 2 & 3 \\
4 & 5 & 6 \\
7 & 8 & 9
\end{bmatrix}
\]

has 9 entries which we can think of as the components of a 2nd order tensor. Each of the components of a 2nd rank tensor, though, has two directions associated with it.

e. Notice that \( a \cdot b \) is a scalar

\[ a \cdot T, \ T \cdot a \] are vectors \( a \cdot T \neq T \cdot a \)

\[ a \cdot T \cdot b, \ b \cdot T \cdot a \] are scalars.

Note: In the expression, \( a \cdot T \cdot b \), the order in which the inner products are taken does not matter.

To see this,

\[ (a \cdot T) \cdot b = (a; T_{ij} e_j) \cdot b_k e_k = a; T_{ij} b_j \quad \left( = \sum_{i=1}^{3} \sum_{j=1}^{3} a_i T_{ij} b_j \right) \]

and

\[ a \cdot (T \cdot b) = a_i e_i \cdot (T_{jk} b_k e_j) = a; T_{ik} b_k \]

\[ \therefore \ a \cdot T \cdot b = a \cdot (T \cdot b) \neq a \cdot b \cdot T \cdot a \]

However, in general, \( a \cdot T \cdot b \neq b \cdot T \cdot a \)
3. 2
th order tensors (continued)

f. The unit tensor \( I \) is defined as

\[
I = \delta_{ij} e_i e_j
\]

and clearly has the property

\[
a \cdot I = a \\
I \cdot b = b
\]

4. Higher order tensors - It is straightforward to construct higher order tensors \( \Rightarrow \) Add indices

a. Here we will only mention third order tensors (or third order dyadics)

\[
S = S_{ijk} e_i e_j e_k
\]

Notice, given the vector \( a = a_1 e_1 \),

\[
a \cdot S = a_1 e_1 \cdot S_{ijk} e_i e_j e_k = a_1 S_{ijk} e_j e_k
\]

Similarly, \( S \cdot a = S_{ijk} e_k e_i e_j \) is a 2nd order tensor

b. Permutation tensor \( \equiv \mathcal{E} \equiv E_{ijk} e_i e_j e_k \)

where \( E_{ijk} \) is the permutation symbol introduced earlier when discussing the vector product.

Exercise: Show \( I \wedge I = -\mathcal{E} \) (be careful with order of vector operations and indices)
5. Symmetric & Anti-symmetric tensors

a. Recall the transpose of a matrix

If \( A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \) then \( A^T = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{12} & A_{12} & A_{13} \\ A_{13} & A_{13} & A_{13} \end{bmatrix} \)

Furthermore, a matrix was called **symmetric** if \( A = A^T \) and was called **anti-symmetric** if \( A = -A^T \)

\[
A = A^T = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{12} & A_{22} & A_{23} \\ A_{13} & A_{23} & A_{33} \end{bmatrix} \quad A = -A^T = \begin{bmatrix} 0 & A_{12} & A_{13} \\ -A_{12} & 0 & A_{23} \\ -A_{13} & -A_{23} & 0 \end{bmatrix}
\]

b. We now generalize these ideas to 2nd rank tensors

(i) A 2nd order tensor \( T \) is **symmetric** if \( T = T^T \) or \( T_{ij} = T_{ji} \)

(ii) A 2nd order tensor \( T \) is **anti-symmetric** if \( T = -T^T \) or \( T_{ij} = -T_{ji} \)

Some important properties (analogous to matrices shown above):

- symmetric tensor \( \Rightarrow T_{ij} = T_{ji} \) \( \Rightarrow \) only 6 independent components
- anti-symmetric tensor \( \Rightarrow T_{ij} = -T_{ji} \) \( \Rightarrow \) only 3 independent components

\( \Rightarrow \) and we will see in the homework that an anti-symmetric tensor can be represented using a vector.

c. Every 2nd order tensor can be expressed as the sum of a symmetric and an anti-symmetric tensor.

\[
(A+B)^T = A^T + B^T
\]

For example,

\[
T = \frac{1}{2} \left( T + T^T \right) + \frac{1}{2} \left( T - T^T \right)
\]

\( \text{symmetric} \quad \text{anti-symmetric} \)

(\text{this is analogous to decomposition of a function into even and odd functions, p. 56})
So, now you have seen that a 2nd order tensor is a mathematical object that (linearly) "sends vectors into other vectors."

This idea is particularly useful when describing physical situations where the physical properties are different in different directions, and you are representing quantities characterized by a vector.

Examples:

(i) Current generated due to an applied electric field $E$.

$$J = \sigma \cdot E$$

- Current density
- Electrical conductivity
- Tensor

(ii) Angular momentum $L$ about a point for a body rotating with angular velocity $\omega$.

$$L = I \cdot \omega$$

- Moment of inertia
- Tensor

Personally, I find (iii) & (iv) the simplest physical situations which suggest the appearance of tensorial quantities in a mathematical description of the physical world.

(iv) Heat flux $\mathcal{G}$ due to a thermal gradient (i.e., a temperature difference).

$$\mathcal{G} = K \cdot \nabla T$$

- Thermal conductivity
- Tensor

A practical example is actually a material like wood. For a given temperature difference (or $\nabla T$), more energy is transferred along the grain than across the grain—the thermal conductivity of the material is different in different directions.

(iv) Stress Tensor — frequently one requires information about stresses (force/area) acting on a material. To describe the state of stress at a point, e.g., $\sigma$, it is necessary to specify the orientation of the surface and the vector forces in two directions are thus necessary. Note: Surface orientation is given by the unit normal vector.
6. Transformation Rules for Tensors

We now examine how the components of a tensor change if we change from one set of Cartesian base vectors to another.

1. Consider two sets of Cartesian base vectors:

\[ \mathbf{e}_i = \text{"old" set of base vectors} \]

\[ \mathbf{e}'_i = \text{"new" set of base vectors} \]

2. First, recall the transformation rule for the components of a vector

(remember that a vector, say \( \mathbf{a} \), is invariant with respect to a change of coordinate system. However, the representation of the vector in terms of its components clearly depends on the choice of the coordinate system.)

\[ \mathbf{a}' = a'_i \mathbf{e}'_i = \mathbf{q} \cdot \mathbf{e}' \]

3. The components \( a_i \) and \( a'_i \) are given by

\[ a'_i = \mathbf{q} \cdot \mathbf{e}'_i = a_i \mathbf{q}' \cdot \mathbf{e}_i \]

4. Definition: the **direction cosines** between the axes are given by

\[ l_{ij} = \mathbf{e}_i \cdot \mathbf{e}_j \]

**Note:** \( l_{mn} \neq l_{nm} \)

**Substitution:**

\[ (\mathbf{e}_i \cdot \mathbf{e}_j) \mathbf{q} \cdot \mathbf{e}_i = a'_i \mathbf{e}_i \]

Rearranging indices allows one to conclude

\[ (\mathbf{e}_i \cdot \mathbf{e}_j) \mathbf{q} = \mathbf{e}_i' \]

Exercise: Demonstrate these relations for yourself.
d. Properties of the $L_{ij}$

As shown on the previous page, $e_i = L_{ij} e_j$, $e_i = L_{ij} e_j$

So, since the base vectors in each of the coordinate systems are orthogonal,

\[
\bar{e}_i \cdot \bar{e}_j = \delta_{ij} \Rightarrow (L_{ij} e_j) \cdot (L_{mn} e_n) = \delta_{ij}
\]

\[
\Rightarrow \quad L_{ij} L_{mn} = \delta_{im}
\]

Similarly, $e_i \cdot e_m = \delta_{im} \Rightarrow L_{ik} L_{mk} = \delta_{im}$

Notice: one can use the above 2 eqns in useful ways.

For example, beginning with eqn (2),

\[
q_i = q_j L_{ij} \Rightarrow \quad q_k L_{ki} = a j L_{ij} L_{ki} \quad \text{after multiplying both sides by } L_{ki}
\]

\[
= a_k \Rightarrow \quad \text{which is the same result as eqn (3)}
\]

h. The above illustrates the details of how the components of a vector transform when a different Cartesian coordinate system is considered.

e. Now, consider the transformation rule for 2\(^{nd}\) order tensors

Begin with $e_i = L_{ij} e_j$, $e_i = L_{ij} e_j$ \quad \text{we know how.}

The 2\(^{nd}\) order tensor $T$ can then be represented as

\[
T = T_{ij} e_i e_j = T_{ij} (L_{kj} e_k)(L_{jn} e_n) = T_{ij} L_{kj} L_{jn} e_k e_n = T'_{km} e_k e_n
\]

\[
\Rightarrow \quad T'_{km} = T_{ij} L_{kj} L_{jn}
\]

Transformation rule for the components of $T$ in the $^\prime$ coordinate system ($e'_i$) relative to the original coordinate system.

Similarly, $T = T_{ij} e_i e_j = T_{ij} (L_{kj} e_k)(L_{jn} e_n) = T_{ij} L_{kj} L_{jn} e_k e_n = T_{km} L_{km} e_i e_j$

\[
\Rightarrow \quad T_{km} = T_{ij} L_{kj} L_{jn}
\]

Transformation rule from "new" to "old"

You may wish to try this as an exercise.

Again, one can do the same for third order tensors $S_{ijk} e_i e_j e_k = S_{ij} L_{ik} L_{jk} L_{kr}$

\[
S'_{ijk} = S_{ij} L_{ik} L_{jk} L_{kr}
\]

and one finds $S'_{ij} = S_{ij} L_{ik} L_{jk} L_{kr}$ and $S'_{ijk} = S_{ijk} L_{ik} L_{jk} L_{kr}$
7. Remark 2. The "definition" of 2nd order tensors

On the previous page, we presented formulas which show how the components of a tensor are related in different Cartesian coordinate systems.

\[ T'_{kj} = T_{ij} l^i_k l^j_m \]
\[ T_{km} = T_{ij} l^i_k l^j_m \]

where

\[ I = T_{km} e^k e^m = T'_{km} e'^k e'^m \]

Hence, given the components of \( T_{ij} \) relative to the \( e_i \)-axes, the above transformation rules specify the components \( T'_{ij} \) in any other set of right-hand Cartesian orthonormal base vectors, \( e'_i \).

Classical treatments of tensor often define a second order tensor as that entity whose components in any, and every, two sets of Cartesian axes transform according to the above rules. The two approaches (the first illustrated on pages 106-107 or the alternative which simply defines a tensor via the above transformation rules) are equivalent, and we will call any entity whose components transform according to these rules a second order tensor.

It is nevertheless important to keep in mind that the tensor itself is defined independent of the choice of coordinate system, but the representation in terms of components depends on the choice of the coordinate system.

An example of this:

Suppose the "components" \( T_{ij} \) are given for all Cartesian axes. Also suppose for all vectors \( a = q_i e_i \) that \( a_i T_{ij} e_j \) is a vector. Then, \( I = T_{ij} e_i e_j \) (that quantity constructed from the components \( T_{ij} \)) must be a second order tensor.

Proof: Use the above transformation which specify how the components of a second order tensor are related.

\[ q_i T_{ij} e_j = (q_i T_{km} l^i_k l^j_m) e^m \]

and since \( q_i \) is a vector

\[ a_k = a_p l^k_p \] or \[ q_i T_{ij} = a_i l^i_k T_{km} l^j_m \]

or

\[ q_i (T_{ij} - T_{km} l^i_k l^j_m) = 0 \] for all vector \( a \)

\[ T_{ij} = T_{km} l^i_k l^j_m \] which is the transformation rule for the components of a second order tensor.
8. Isotropic tensors

a. Definition: Any tensor which has the same components in all Cartesian axes is called an isotropic tensor.

Now, on the previous two pages, we examined how to represent the components of vectors and tensors for different Cartesian coordinate axes.

b. Recall the unit tensor $I = \delta_{ij}e_i e_j$.

What are the components of $I = \delta_{ij}$, relative to the $e_i$ axes.

Well, from pg. 107,

$\delta_{ij}' = \delta_{mn} L_{mi} L_{nj} = L_{ni} L_{nj} = \delta_{ij}'$ via the identification on the top of p. 92.

$c. I$ has the same components ($\delta_{ij}$) in all Cartesian systems, and hence $I$ is called an isotropic 2nd order tensor.

$\Rightarrow$ It can be shown that apart from a multiplicative constant, the unit tensor $I$ is the only isotropic 2nd order tensor.

c. Likewise, consider the permutation tensor $\varepsilon = E_{ijk} e_i e_j e_k$

$E = E_{ijk} e_i e_j e_k = E'_{ijk} e_i e_j e_k$ in the primed coordinate system.

First, a closer observation:

$E_{ijk} = \varepsilon_i \cdot (e_j \wedge e_k)$

From the bottom of p. 107, we see that the components $E'_{iqr}$ and $E_{ijk}$ are related by

$E'_{iqr} = E_{ijk} \varepsilon_{ip} e_j \wedge e_k = \varepsilon_i \cdot (e_j \wedge e_k) \varepsilon_{ip} e_j \wedge e_k$

$= (\varepsilon_{ip} \varepsilon_i) \cdot [e_j \wedge e_k \wedge e_r e_k]$

$= \varepsilon_{ip} \cdot (e_j \wedge e_r) = E_{pqr}$ simply because $\varepsilon_{ip} \varepsilon_i e_r$ form a right-handed orthogonal coordinate system.

$\therefore E$ is a third order isotropic tensor.
9. Example: a change of cartesian axes \( f_i' = f_{m} l_{mi} \)

let \( \frac{\partial f_i}{\partial x_j} = f_i' l_{ji} \) be a vector field.

Consider the derivatives of the \( f_i \) and \( f_i' \) with respect to position, i.e., consider

\[
\frac{\partial f_i'}{\partial x_j} = \frac{\partial (f_m l_{mi})}{\partial x_k} \frac{\partial x_k}{\partial x_j}
\]

and we know (p. 91) that the components of the position vector

satisfy

\[
x_i' = x_j' \delta_{ji} \quad x_i = x_j' \delta_{ij} \quad \Rightarrow \frac{\partial x_i}{\partial x_j} = \delta_{ij}
\]

Clearly, via the chain rule,

\[
\frac{\partial f_i'}{\partial x_j} = \delta_{ij} l_{kji} l_{mi}
\]

which is, the same transformation rule found on p. 107 for the

components of a 2nd order tensor.

Therefore, \( \frac{\partial f_i'}{\partial x_k} \) (or \( \frac{\partial f_i'}{\partial x_j} \)) represents the components of

a 2nd order tensor.

The notation generally used is \( \smallint f_i' = \frac{\partial f_i'}{\partial x_i} e_i e_j \)

9. Scalar Invariants - scalars simply have a magnitude

a. Example: consider the scalar \( a \cdot b \)

\( a \cdot b = a_i b_i' = a_i' b_i \) and this scalar quantity is an invariant, in other words, it has the same

value in all Cartesian axes.

To demonstrate this use the transformation rules discussed earlier.

\( a_i b_i' = a_j l_{ji} b_k l_{ki} = a_k b_k \)

b. Example: if \( T_{ij} = T_{ji} \) is a second order tensor, then

\( T_{ii} \) is an invariant.

Proof: \( T_{ij} = T_{km} l_{ki} l_{mj} \Rightarrow T_{ii} = T_{km} l_{ki} l_{mi} = T_{kk} \)

\( T_{kk} = \text{Trace} (T) \equiv tr T \)
c. Invariance of the divergence of a vector \( \mathbf{v} \cdot \mathbf{f} \).

Since \( \mathbf{v} \cdot \mathbf{f} = \frac{df_i}{dx_j} \) and since \( \frac{df_i}{dx_j} \) are the components of a second order tensor, say \( T = T_{ij} e_i e_j \), then it follows that because we just saw that \( T_{ii} \) is an invariant, it also must be true that \( T_{ii} = \frac{df_i}{dx_i} \) is an invariant.

Remark: SOME ADDITIONAL NOTATION

The "Double Dot product":

The following notation is sometimes used:

\[ \mathbf{a} \cdot \mathbf{b} = \mathbf{c} \cdot \mathbf{d} = (\mathbf{a} \cdot \mathbf{d})(\mathbf{b} \cdot \mathbf{c}) \rightarrow \text{a scalar} \]

Also, for two second order tensors, \( \mathbf{T}, \mathbf{S} \)

\[ \mathbf{T} : \mathbf{S} = T_{ij} S_{ij} = S_{kl} T_{kl} = T_{ij} S_{kl} \delta_{il} \delta_{jk} = T_{ii} S_{ii} \]

Note: nearest two indices are the same.

So

\[ \mathbf{T} : \mathbf{T} = T_{ii} = tr(\mathbf{T}) \]

trace of the second order tensor; by summation convention,

\[ tr \mathbf{T} = T_{11} + T_{22} + T_{33} \] (See bottom of p. 110)
10. EXAMPLE: The Stress Tensor

Fundamental to the description and a basic understanding of the deformation of solids and the flow of fluids.

a. Why is this concept useful?

Suppose you were interested in the state of equilibrium of a material. Newton's law applied to a small piece of material says
\[ \sum \text{Forces} = \text{mass} \times \text{acceleration}. \]

If the material is stationary and at equilibrium (acceleration = 0), then the sum of the forces on a piece of the material must balance. So, now consider the small cube of material shown below.

We have written
\[ t_{(i)} = \sigma_{ij} \varepsilon_j \]

force/area on the face \( \perp \varepsilon_i \)

the component of the force/area acting in the \( \varepsilon_j \) direction on the face \( \perp \varepsilon_i \).

In general,
\[ t_{(i)} = \sigma_{ij} \varepsilon_j \rightarrow \text{"stress vector" = force/area on face } \perp \varepsilon_i \]

b. Furthermore, there is a very beautiful result due to Cauchy which considers a tetrahedron-shaped material volume and applies the principle that the sum of the forces must balance.

For all \( n \)
\[ t_{(n)} \rightarrow \text{stress vector (force/area) on surface with normal } n \]

At equilibrium: net force on a small volume of material (no motion)
\[ \sum_{\text{volume of material}} \pm \int t_{(n)} \, dS = \int n \cdot \sigma \, dS = 0 \]

(no acceleration)

Body

(condition for static equilibrium)
C. Equilibrium of a continuous material (continued)

We begin with

\[ \iint_S n \cdot \mathbf{G} \, dS = 0 \]

and since this must hold for all volume elements \( V \), we conclude

\[ \nabla \cdot \mathbf{G} = 0 \]

or \( \frac{\partial G_{ij}}{\partial x_i} = 0 \hspace{1cm} \) (for \( i \neq j \))

If you were to write this out, \( \nabla \cdot \mathbf{G} \) is a vector so \( \nabla \cdot \mathbf{G} = 0 \) represents 3 eqns, one for each component of the vector.

So,

\[ \frac{\partial G_{11}}{\partial x_1} + \frac{\partial G_{12}}{\partial x_2} + \frac{\partial G_{13}}{\partial x_3} = 0 \]

\[ \frac{\partial G_{21}}{\partial x_1} + \frac{\partial G_{22}}{\partial x_2} + \frac{\partial G_{23}}{\partial x_3} = 0 \]

\[ \frac{\partial G_{31}}{\partial x_1} + \frac{\partial G_{32}}{\partial x_2} + \frac{\partial G_{33}}{\partial x_3} = 0 \]

Remark: In order to proceed further, you must relate the stress tensor to the small displacements that occur in the material.
II. The Moment of Inertia Tensor

1. In the study of the mechanics of rotating rigid bodies, it is necessary to know the angular momentum and kinetic energy of the object. We will now see how a 2nd order tensor, the moment of inertia tensor, naturally arises.

6. Consider a rigid body spinning with angular velocity \( \omega \) about a fixed point \( O \).

Recall that for a point mass rotating at angular velocity \( \omega \) about some origin, the radial velocity is

\[
\mathbf{v} = \omega \times \mathbf{x}
\]

and the angular momentum about \( O \) is \((\text{mass}) \cdot (\mathbf{x} \times \mathbf{v})\).

So, the total angular momentum \( \mathbf{L} \) of the rigid body is

\[
\mathbf{L} = \int \mathbf{r} \times (\rho \mathbf{v}) \, dV = \int \rho \mathbf{x} \times (\omega \times \mathbf{x}) \, dV
\]

where every point of the rigid body rotates with angular velocity \( \omega \).

But,

\[
\mathbf{x} \times (\omega \times \mathbf{x}) = (\mathbf{x} \times \mathbf{x}) \omega - \mathbf{x} (\mathbf{x} \cdot \omega)
\]

\[
= [\mathbf{x} \times \mathbf{x}] \omega - \mathbf{x} (\mathbf{x} \cdot \omega)
\]

Exercise: prove this identity.

So,

\[
\mathbf{L} = \int \left[ (\mathbf{r}^2 \mathbf{I} - \mathbf{x} \mathbf{x}) \right] \omega \, dV
\]

or since \( \omega \) is constant throughout \( V \)

\[
\mathbf{L} = \int (\mathbf{r}^2 \mathbf{I} - \mathbf{x} \mathbf{x}) \rho \, dV \cdot \omega = \frac{\mathbf{I}}{\omega} \cdot \omega
\]

where \( \mathbf{I} \) is symmetric

\[
\mathbf{I}(\mathbf{x}) = \int (\mathbf{r}^2 \mathbf{I} - \mathbf{x} \mathbf{x}) \rho \, dV = \text{moment of inertia tensor about } O.
\]