Lecture 4. Interfacial boundary conditions

But first, back to puddles...

**Total Energy**:  
\[ E = (\gamma_s - \gamma_{sv}) A + \gamma_{sv} A + \frac{1}{2} \rho g h^2 A \]

\[ = -S \frac{V}{h} + \frac{1}{2} \rho g V h \]

**Minimum Energy**:  
\[ \frac{dE}{dh} = 0 = S \frac{V}{h^2} + \frac{1}{2} \rho g V \]

when \(-S \frac{1}{h^2} = \frac{1}{2} \rho g\)

\[ h_0 = \left(-\frac{2S}{\rho g}\right)^\frac{1}{2} \]

\[ = 2 \ell_c \sin \frac{\theta_e}{2} \]
Capillary Adhesion:

Two wetted surfaces can stick together with great strength if \( \theta_e < \frac{\pi}{2} \).

E.g., two glass plates with Si oil between them.

Laplace Pressure:

\[
\Delta P = \sigma \left( \frac{1}{R} - \frac{\cos \theta_e}{H/2} \right) \approx -\frac{2\sigma \cos \theta_e}{H}
\]

I.e., low \( P \) inside film provided \( \theta_e < \frac{\pi}{2} \).

If \( H \ll R \), \( F = \pi R^2 \frac{2\sigma \cos \theta_e}{H} \)

Is the attractive force between the two plates.

E.g., for \( H_2O \), with \( R = 1 \text{ cm} \), \( H = 5 \text{ mm} \) and \( \theta_e = 0 \), one finds \( \Delta P \approx \frac{1}{3} \text{ atm} \) and adhesive force \( F \approx 10 \text{ N} \), the wt of 1L of \( H_2O \).

\( \Rightarrow \) this is used by beetles in nature.

Q: How do they deadlheve?
Interfacial Fluid Mechanics

Governing Eqs: Navier Stokes

For an incompressible, homogeneous fluid of density $\rho$ and viscosity $\mu = \rho \nu$, acted upon by an external force per unit volume $\mathbf{f}$. Its velocity $\mathbf{u}$ and $p$ fields evolve according to:

$$\rho \left( \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} \right) = -\nabla p + \mathbf{f} + \mu \nabla^2 \mathbf{u}$$

$$\nabla \cdot \mathbf{u} = 0 \quad \frac{\partial \mathbf{u}}{\partial t} : \text{MATERIAL/LAGRANGIAN DERIVATIVE}$$

This system of 4 equations in 4 unknowns ($u_1, u_2, u_3, p$) must be solved subject to appropriate BCs.

Fluid-Solid BCs: no-slip: $\mathbf{u} = \mathbf{u}_{solid}$

Eq. 1 Falling sphere: $\mathbf{u} = \mathbf{V}$ on $S$

Eq. 2 Convection in a box: $\mathbf{u} = 0$ on $S'$

COLD

HOT
In this course, we are concerned with flows dominated by interfacial effects, e.g., drop motion, e.g., water waves.

Note: these interfaces are free to move; thus, this type of problems are known as FREE BOUNDARY PROBLEMS.

**Continuity of Velocity** at an interface requires that

\[ \mathbf{U} = \hat{\mathbf{U}}. \]

And what about pressure?

We've seen that \( \Delta P \sim \sigma/R \) for a static bubble/drop. But to answer this question in general, we must develop stress conditions at a fluid–fluid interface.
Recall: Stress Tensor

For an incompressible Newtonian fluid, the state of stress within a fluid is described by the stress tensor:

\[ T = -p \mathbb{I} + 2\mu \mathcal{E} \]

where \[ \mathbb{I} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \]

\[ \mathcal{E} = \text{viscous stress} \]

where \[ \mathcal{E} = \frac{1}{2} \left[ \nabla \mathbf{u} + (\nabla \mathbf{u})^T \right] \]

is the deviatoric stress tensor.

In Cartesian coords \((x, y, z)\), we can expand as

\[ T = \begin{pmatrix} -p + 2\mu \frac{\partial u_1}{\partial x_1} & M \left( \frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} \right) & M \left( \frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1} \right) \\ M \left( \frac{\partial u_2}{\partial x_1} + \frac{\partial u_1}{\partial x_2} \right) & -p + 2\mu \frac{\partial u_2}{\partial x_2} & M \left( \frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2} \right) \\ M \left( \frac{\partial u_3}{\partial x_1} + \frac{\partial u_1}{\partial x_3} \right) & M \left( \frac{\partial u_3}{\partial x_2} + \frac{\partial u_2}{\partial x_3} \right) & -p + 2\mu \frac{\partial u_3}{\partial x_3} \end{pmatrix} \]

The associated hydrodynamic force per unit volume (force density) within a fluid is \( \mathbf{D} \cdot \mathbf{T} \).

Navier–Stokes:

\[ \rho \frac{D\mathbf{u}}{Dt} = \mathbf{D} \cdot \mathbf{T} + \mathbf{f} \]

\[ = -\nabla p + \mu \nabla^2 \mathbf{u} + \mathbf{f} \]
Now \( T_{ij} = \frac{\text{force}}{\text{area}} \) acting in the \( \hat{e}_i \) direction on a surface whose normal is \( \hat{e}_j \).

**Note:**
1. **Normal stresses (diagonals)** \( T_{11}, T_{22}, T_{33} \) involve both \( P \) and \( u_i \).
2. **Tangential stresses (off-diagonals)** \( T_{12}, T_{13}, T_{23}, \text{etc.} \) involve only velocity gradients, i.e., viscous stresses.
3. \( T_{ij} \) is symmetric.
4. \( \mathbf{T}(\mathbf{n}) = \mathbf{n} \cdot \mathbf{T} = \text{stress vector acting on surface with normal } \mathbf{n} \).

**Example:** Boundary stress from a shear flow is tangential force/area on lower surface (in \( x \)-direction on surface with \( \mathbf{n} = \hat{e}_x \))

\[
T_{yx} = \mu \frac{\partial u_x}{\partial y} = \mu k
\]

**Note:** Form of \( \mathbf{T} \) in arbitrary curvilinear coordinates is given in the Appendix of Batchelor.
Interfacial Boundary Conditions

Consider an interfacial surface bounded by a closed curve $C$.

where $\hat{n}$ is unit normal to $S$, $C$

$\Sigma$ is unit normal to $C$, tangent to $S$

$m$ is unit tangent to $S$, $C$

Consider an infinitesimal cylindrical pillbox $V$ of radius $\Sigma$ and height $\xi$, (i.e., $\xi \ll \Sigma$) that intersects $C$.

For a force balance on $V$, we must consider:

$\hat{t}(\hat{n}) = \hat{n} \cdot \hat{T}$ is the force/area exerted by upper fluid ($+$)

$\hat{t}(\hat{\nu}) = \hat{\nu} \cdot \hat{T}$ is """" lower fluid ($-$)

Force balance:
\[ \int_V \frac{D\mathbf{u}}{Dt} \, dV = \int_T \mathbf{T} \cdot d\mathbf{V} + \int_S \left[ \mathbf{f}(\mathbf{n}) + \hat{\mathbf{t}}(\mathbf{n}) \right] \, dS + \oint_S \sigma \cdot d\mathbf{l} \]

- Inertial force
- Body forces
- Hydrodynamic forces acting on upper and lower surfaces
- Surface tension

Now, the inertial + body forces must scale as \( \Sigma^2 \Sigma \), while the surface forces scale as \( \Sigma^2 \). Hence, in the limit of \( \Sigma, \rightarrow 0 \), surface forces must balance:

\[ \int_S \mathbf{f}(\mathbf{n}) + \hat{\mathbf{t}}(\mathbf{n}) \, dS + \oint_S \sigma \cdot d\mathbf{l} = 0 \]

Now, we have \( \mathbf{f}(\mathbf{n}) = \mathbf{n} \cdot \mathbf{T} \), \( \hat{\mathbf{t}}(\mathbf{n}) = \mathbf{n} \cdot \mathbf{T} \):

\[ \mathbf{n} \cdot \mathbf{T} = -\mathbf{n} \cdot \mathbf{T} \]

Moreover, application of Stokes' Theorem (see Handout 1) allows us to write:

\[ \int_S \sigma \cdot d\mathbf{l} = \int_S \nabla_S \sigma - \sigma \mathbf{n} (\nabla_S \cdot \mathbf{n}) \, dS \]

where \( \nabla_S \equiv (\mathbf{I} - \mathbf{n} \mathbf{n}) \cdot \mathbf{D} = \mathbf{D} - \mathbf{n} \frac{\partial}{\partial \mathbf{n}} \)

is the tangential gradient operator,
required because $\sigma$ and $\mathbb{N}$ are only defined on the interface. We proceed by dropping the subscript, $D_s \to D$, with this understanding.

The surface force balance:

$$\oint (n \cdot \vec{T} - \vec{u} \cdot \hat{\vec{T}}) \, ds' = \oint \sigma_{\perp} (\nabla \cdot n) - D_0 \, ds$$

Now since the surface element $ds'$ is arbitrary, the integrand must vanish identically.

$\Rightarrow$ **Intertitial Stress Balance Equation**

$$n \cdot \vec{T} - \vec{u} \cdot \hat{\vec{T}} = \sigma_{\perp} (\nabla \cdot n) - D_0 \sigma$$