The motion of an inviscid drop in a bounded rotating fluid

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The motion of a buoyant inviscid drop rising vertically along the rotation axis of a rapidly rotating low viscosity fluid bounded above and below by rigid horizontal boundaries is considered in the case that the drop is circumscribed by a Taylor column that spans the entire fluid depth. Both the shape and steady rise speed of the drop are deduced as a function of the interfacial tension. The analysis demonstrates that the drop assumes the form of the prolate ellipsoidal figure of revolution which would arise in the absence of any relative motion in the surrounding fluid. The hydrodynamic drag on the drop follows simply from the analysis of Moore and Saffman [J. Fluid Mech. **31**, 635 (1968)], who considered the equivalent motion of a rigid particle. The rise speed of a deformed inviscid drop is approximately one-half that of an identically shaped rigid particle; in particular, the rise speed of a spherical inviscid drop is 0.41 that of a rigid sphere.

I. INTRODUCTION

The motion and deformation of drops in rotating fluids arises in a variety of applications, including industrial separations, materials processing, and geophysical fluid dynamics. We describe herein the buoyancy-driven motion of a deformable inviscid drop translating along the axis of a rapidly rotating fluid of low viscosity, bounded above and below by rigid horizontal boundaries. In particular, we bring together two studies: (1) an extension of the analysis originally presented by Moore and Saffman[1] for the drag on a rigid particle rising along the length of a Taylor column; and (2) the well-known calculation for the shape of a stable, rigidly rotating drop held together by surface tension.[2–6] Our analysis yields a simple prediction for the shape and steady rise speed of an inviscid drop in the case that both inertial and viscous effects are negligible, so that a "geostrophic balance" exists in the bulk of the surrounding fluid.

The Taylor–Proudman theorem requires that all fluid motions in a geostrophically balanced incompressible flow be independent of the spatial coordinate that varies in a direction parallel to the axis of rotation. Consequently, when a body moves slowly through a fluid rotating rapidly about a vertical axis, it drags along with it a vertical column of fluid, in which there is no vertical motion relative to the body.[7] The Taylor column that circumscribes the body has a vertical extent determined by the fluid viscosity, and will span the entire depth of a sufficiently shallow horizontal fluid layer. In this case, as the body rises, the boundaries of the layer act to disrupt the vertical motion of the fluid in the Taylor column. A qualitative representation of the resulting flow field is illustrated in Fig. 1 for the case of a stress-free body rising between rigid horizontal boundaries.

A physical description of the detailed fluid motion within the bounded Taylor column is best given in terms of vorticity dynamics. The vertical vorticity field in the fluid associated with the fluid's solid body rotation is perturbed by the rise of the body (see Fig. 1). Vortex compression upstream of the body generates negative relative (vertical) vorticity, while vortex stretching in the body's wake generates positive relative vorticity. The associated swirl velocities couple to the rigid container boundaries through the action of viscosity in thin boundary layers, giving rise to Ekman pumping and suction on, respectively, the upper and lower container boundaries. If the translating particle is rigid, an additional thin Ekman boundary layer is attached to the particle surface; however, in the limit that the particle surface is stress-free, which approximates an inviscid drop or bubble, no such boundary layer exists. The flow picture is completed by continuity, which requires a net downflow in thin Stewartson layers that define the vertical walls of the Taylor column.[8] The role of the fluid viscosity is to relax the Taylor–Proudman constraint of two-dimensionality through the generation of small-scale boundary layer motions: the body is able to rise only by virtue of the Ekman transport from the fore to the aft regions of the Taylor column.

The problem of particle translation along the length of a vertical Taylor column spanning the entire depth of a bounded horizontal fluid layer was first considered by Moore and Saffman.[1,8] An accompanying experimental study was performed by Maxworthy.[9] Of the various possible combinations of rigid and/or stress-free particle and container boundaries, three were considered by Moore and Saffman:[1] (i) a rigid particle between rigid boundaries; (ii) a rigid particle between a stress-free upper boundary and rigid lower boundary; and (iii) a particle with a stress-free surface (e.g., a bubble) between stress-free boundaries.

For a rigid sphere of radius a rising vertically between rigid boundaries through a fluid of density ρ, kinematic viscosity ν, and rotating with angular speed Ω, Moore and Saffman determined the steady rise speed U_{rr} to be[10]

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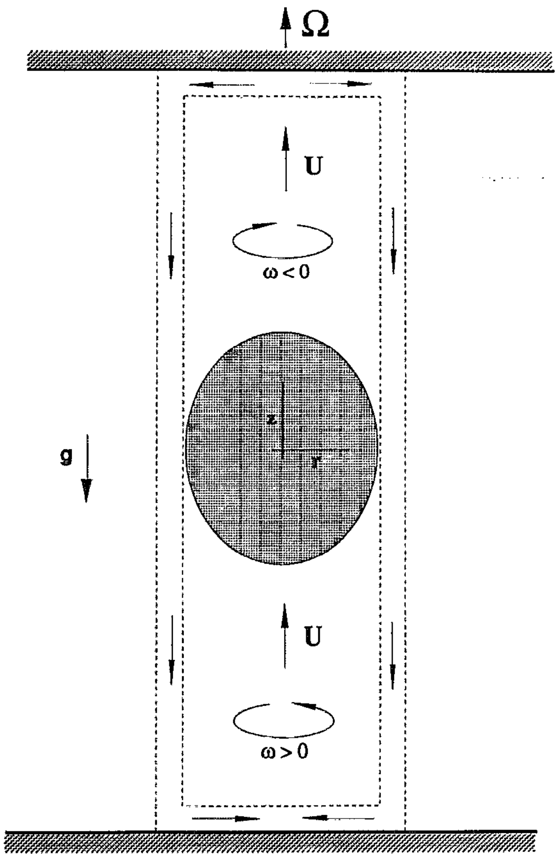

FIG. 1. A schematic illustration of the flow induced by a stress-free axisymmetric body rising through a rapidly rotating low viscosity fluid bound by rigid horizontal boundaries.

$$U_{rr} = \frac{140}{43} \frac{\Delta\rho}{\rho} \frac{g}{\Omega a} \sqrt{\frac{\nu}{\Omega}}, \tag{1}$$

where $\Delta\rho$ is the density difference between the sphere and the fluid. In this case, since both the particle and container boundaries are rigid, Ekman transport occurs over both surfaces. For the case of a rigid sphere rising between stress-free container boundaries, the rise speed U_{rf} decreases to $U_{rf} = \frac{15}{8} \times (\Delta\rho\, g)/(\rho\Omega a)\sqrt{\nu/\Omega} \approx 0.58\, U_{rr}$. Since Ekman layers cannot develop on the stress-free container boundaries, the fluid can be transported from the upstream to the downstream regions of the Taylor column only by way of the Ekman layers on the body surface. The decreased efficiency of this fluid transport mechanism is responsible for the decreased rise speed of the rigid sphere. Finally, Moore and Saffman state that a stress-free buoyant particle cannot rise between stress-free boundaries since, in this case, Ekman layers cannot develop on either the body or container boundaries, and thus fluid cannot be transported from the upstream to the downstream regions of the Taylor column. As the authors indicate, this paradoxical result suggests the inadequacies of the geostrophic approximation in describing this particular flow.

In this paper, we extend the aforementioned work by considering a deformable inviscid drop, which represents the limit of a *stress-free* body, rising between rigid horizontal boundaries in the case that the associated Taylor column spans the entire fluid depth. In Sec. II, we determine the form of the flow induced by a rising stress-free body with a prescribed axisymmetric shape. Our analysis, which closely follows that of Moore and Saffman,[1] leads naturally to the calculation of the body's hydrodynamic drag and steady rise speed presented in Sec. III. In Sec. IV, using approximations consistent with those invoked in the drag calculation, we deduce the shape of a deformable inviscid fluid drop bound by interfacial tension. The drop shape is determined from a balance between centrifugal forces and the interfacial tension forces of the curved interface. The resulting equilibrium axisymmetric shapes correspond to the well-known family of prolate ellipsoids formed by fluid drops suspended in a rapidly rotating fluid of higher density.[3-6] Finally, since the drop has an axisymmetric shape, the drag result of Sec. III may be applied directly, and so we are able to determine both the drop shape and steady rise speed as a function of the interfacial tension.

II. DYNAMICAL PICTURE

We begin with a description of the dominant physical processes accompanying particle translation through rapidly rotating fluids. While our discussion differs only slightly from that of Moore and Saffman,[1] we include it here in order to illustrate the self-consistency of the approximations made in the drag and drop shape calculations.

Consider a plane layer of incompressible fluid contained above and below by rigid horizontal boundaries. The system rotates about a vertical axis with constant angular velocity $\mathbf{\Omega}$ in the presence of a vertical gravitational field $\mathbf{g}$. The solid body rotation of the fluid is disrupted by the slow, steady, on-axis rise of a buoyant inviscid drop. We introduce a cylindrical coordinate system (r,θ,z) with origin at the drop's center of mass and with the z axis vertical, so that $\mathbf{\Omega} = \Omega\,\hat{\mathbf{z}}$ and $\mathbf{g} = -g\,\hat{\mathbf{z}}$. Henceforth, the superscripts "+" and "−" denote flow variables in the upstream $(z>0)$ and downstream $(z<0)$ regions of the fluid, respectively. The drop is assumed to be axisymmetric, with a steady shape specified by $z = f^{\pm}(r)$ for $r<R$, where R is the "equatorial" radius of the drop. In Sec. IV we demonstrate that, in the dynamic limit considered in this paper, a deformable drop will assume a shape that is not only axisymmetric, but fore–aft symmetric, so that $f^{+}(r) = -f^{-}(r)$.

In a frame rotating uniformly with the container, the fluid velocity $\mathbf{v}(\mathbf{r}) = (u,v,w)$ is related to that in the stationary frame, $\mathbf{u}(\mathbf{r})$, by $\mathbf{v}(\mathbf{r}) = \mathbf{u}(\mathbf{r}) - \mathbf{\Omega}\wedge\mathbf{r}$, and the Navier–Stokes equations take the familiar form

$$\frac{\partial\mathbf{v}}{\partial t} + \mathbf{v}\cdot\nabla\mathbf{v} + 2\mathbf{\Omega}\wedge\mathbf{v} = -\frac{1}{\rho}\nabla p_d + \nu\,\nabla^2\mathbf{v}, \tag{2}$$

$$\nabla\cdot\mathbf{v} = 0.$$

The dynamic pressure p_d is related to the fluid pressure p by

$$p_d = p + \rho g z - (\rho/2)\Omega^2 r^2. \tag{3}$$

Taking the curl of (2) yields an equation governing the evolution of the relative vorticity, $\boldsymbol{\omega} = \nabla\wedge\mathbf{v}$, of the fluid,

$$\frac{\partial \boldsymbol{\omega}}{\partial t} + \mathbf{v}\cdot\nabla\boldsymbol{\omega} = (2\boldsymbol{\Omega} + \boldsymbol{\omega})\cdot\nabla\mathbf{v} + \nu\,\nabla^2\boldsymbol{\omega}. \tag{4}$$

The drop's steady rise speed U is assumed to be sufficiently slow, and the rotation rate of the fluid Ω sufficiently rapid that both the particle Rossby and Ekman numbers, respectively, $\mathscr{R}_0 = U/R\Omega$ and $E_k = \nu/\Omega R^2$, are small. A geostrophic flow is thus established in the bulk of the surrounding fluid. Viscous effects are important only within thin Ekman layers on the rigid upper and lower container boundaries. In the limit of small Ekman number, the ratio of the boundary layer thickness $\delta = \sqrt{\nu/\Omega}$ to the characteristic drop dimension is necessarily small: $\delta/R = \sqrt{E_k} \ll 1$.

Boundary layer scaling of the steady vorticity equation reveals that, in the limit $(\mathscr{R}_0,\ E_k,\ \mathscr{R}_0/\sqrt{E_k}) \ll 1$, the diffusion of vertical vorticity ω_z across the upper/lower Ekman layers is balanced by the vortex compression/stretching associated with the vertical velocity gradient across the layers,

$$2\Omega\,\frac{\partial w}{\partial z} \sim \nu\,\frac{\partial^2 \omega_z}{\partial z^2}.$$

The jump in vertical vorticity, $\Delta\omega_z$, across the Ekman layers is thus given approximately by

$$2\Omega\,\frac{U}{\delta} \sim \nu\,\frac{\Delta\omega_z}{\delta^2};$$

that is, $\Delta\omega_z \sim U/\delta$. This implies azimuthal swirling motions within the Taylor column of typical velocity $R\,\Delta\omega_z = U/\sqrt{E_k}$, which are necessarily much larger than the drop's rise speed.

The flow outside of the thin boundary layers is thus characterized by length, velocity, and time scales of, respectively, R, $U/\sqrt{E_k}$, and $R\sqrt{E_k}/U$. Nondimensionalization of Eq. (2) based on this scaling reveals that, in the limit $(\mathscr{R}_0,\ E_k,\ \mathscr{R}_0/\sqrt{E_k}) \ll 1$, inertial and viscous effects may be neglected, so that a geostrophic balance prevails in the bulk of the fluid:

$$2\rho\boldsymbol{\Omega} \wedge \mathbf{v} = -\nabla p_d. \tag{5}$$

Taking the curl of (5) yields the familiar Taylor constraint of two-dimensionality, $\partial\mathbf{v}/\partial z = 0$. The drop is necessarily circumscribed by a Taylor column, in which there is no vertical motion relative to the drop, so that $w = U$. Outside the Taylor column $(r > R)$, the fluid is quiescent. Within the Taylor column, radial pressure gradients are balanced by Coriolis forces associated with azimuthal fluid motion. The geostrophic swirl velocities are formally obtained by application of the Ekman compatibility conditions,[11] which relate them to the vertical velocity, $w^{\pm} = U$, of the fluid in the Taylor column,

$$w^{\pm} = \mp\frac{\delta}{2r}\frac{d(rv^{\pm})}{dr}.$$

Integration reveals that the swirl velocities in the fore and aft regions of the Taylor column are equal in magnitude and opposite in sense of circulation,

$$v^{+}(r) = -v^{-}(r) = -U(r/\delta).$$

The relative vorticity of the fluid is purely vertical and reverses sign in the fore and aft column regions,

$$\omega_z^{\pm} = \mp\Omega\mathscr{R}_0/\sqrt{E_k}.$$

The fluid helicity density H, relative to the rotating frame, is likewise an odd function of z:

$$H^{\pm} = \boldsymbol{\omega}^{\pm}\cdot\mathbf{v}^{\pm} = \mp(\mathscr{R}_0/\sqrt{E_k})\Omega U.$$

Since the swirling motions correspond to simple rigid body rotations, no viscous stresses arise in the bulk flow. Consequently, provided the rising drop is axisymmetric and stress-free, the geostrophic flow will not be disturbed and thus no boundary layer need exist at the drop surface.

The dynamic pressure fields up- and downstream of the drop are the geostrophic-pressures obtained by integrating the radial component of (5), and are given simply by

$$p_d^{\pm}(r) = \pm\rho\Omega U(r^2/\delta). \tag{6}$$

The flow field is now completely described.

III. DRAG CALCULATION

In order for a body of density $\rho_d = \rho - \Delta\rho$ and volume V to rise at a steady rate, the buoyancy force $Vg\,\Delta\rho$ must be balanced by the hydrodynamic drag force. The hydrodynamic drag on the inviscid drop considered in Sec. II, D_{fr}, is obtained by integrating the dynamic pressure p_d over its surface,

$$D_{fr} = \int_S p_d\,\hat{\mathbf{z}}\cdot\mathbf{n}\,dS, \tag{7}$$

where $\mathbf{n}$ is the unit normal to the local body surface S. The vertical drag force on the axisymmetric drop is a result of the difference in geostrophic pressures upstream and downstream of the drop, so that (7) reduces to

$$D_{fr} = \int_0^R 2\pi r\,\{p_d^{+}[r,f^{+}(r)] - p_d^{-}[r,f^{-}(r)]\}dr.$$

Using (6) for the geostrophic pressure field, we obtain by direct integration the result

$$D_{fr} = \pi\rho\Omega U\sqrt{\Omega/\nu}R^4. \tag{8}$$

The drop's steady rise speed U_{fr} is obtained from a balance of buoyancy and drag forces, which yields

$$U_{fr} = \frac{V}{\pi R^4}\frac{\Delta\rho}{\rho}\frac{g}{\Omega}\sqrt{\frac{\nu}{\Omega}}. \tag{9}$$

It follows that the rise speed of a stress-free body of equatorial radius R decreases with rotation rate Ω and increases with the viscosity ν of the surrounding fluid. Again, viscosity acts to relax the Taylor–Proudman constraint of two-dimensional motion imposed by the fluid's rotation.

The axisymmetric drop shape affects the rise speed only insofar as it defines the radius R of the Taylor column. This result is in contrast to that of Moore and Saffman, who demonstrated that the rise speed of an axisymmetric rigid body depends explicitly on the body's geometrical form. Comparing Eqs. (9) and (1) reveals that the rise speed of an inviscid spherical drop is less than half that of a rigid sphere of the same radius: $U_{fr} = 43/105\,U_{rr}$. For an inviscid drop, Ekman layers develop only on the container walls and not on the drop surface, so the fluid in the upstream region of the Taylor column has only one route by which to exit into the downstream region. The Ekman transport mechanism is thus less efficient for a drop than for a rigid body, and the associated rise speed is decreased.

IV. DEFORMABLE INVISCID DROP: SHAPE CALCULATION

In this section we consider the shape of a deformable inviscid fluid drop bound by a constant interfacial tension σ and rising along the fluid's axis of rotation. The steady inviscid drop shape is determined from the normal stress balance at the fluid–fluid interface, which requires that the pressure difference across the interface be balanced by the interfacial tension forces of the curved surface. Thus along the drop's surface S, we have

$$p_{\mathrm{drop}}(r,z) - p^{\pm}(r,z) = \sigma \nabla_s \cdot \mathbf{n}, \tag{10}$$

where $\nabla_s \cdot \mathbf{n}$ is the local surface curvature. If the drop fluid rotates uniformly on axis with the angular velocity $\mathbf{\Omega}$ of the surrounding fluid, the pressure inside the drop is

$$p_{\mathrm{drop}}(r,z) = p_C + \tfrac{1}{2}\Omega^2 \rho_d r^2 - \rho_d g z,$$

where p_C is a reference pressure at the drop's center. In the geostrophic regions up- $(+)$ and downstream $(-)$ of the drop, Eqs. (3) and (6) yield

$$p^{\pm}(r,z) = p_A \pm \rho \Omega U \frac{r^2}{\delta} + \frac{1}{2}\Omega^2 \rho r^2 - \rho g z,$$

where p_A is the pressure that would prevail at the origin in the absence of the drop. Substituting into (10) thus yields

$$p_0 - \Delta\rho \frac{\Omega^2}{2} r^2 + \Delta\rho\, g z \mp \rho \Omega U \frac{r^2}{\delta} = \sigma \nabla_s \cdot \mathbf{n}, \tag{11}$$

where $p_0 = p_C - p_A$. Note that both the third and fourth terms on the left-hand side of (11), which correspond to the contributions of, respectively, the hydrostatic and geostrophic pressures, reverse sign at the equatorial plane $z=0$. These pressures act to destroy the fore–aft symmetry of the drop. Nondimensionalizing all lengths with respect to R (r and z henceforth denote dimensionless distances) and using (9) for the rise speed U reduces (11) to the dimensionless form:

$$P + 4\Sigma r^2 + \mathscr{G} z \mp \frac{4}{3} \mathscr{G} \left(\frac{a}{R}\right)^3 r^2 = \nabla_s \cdot \mathbf{n},$$

where $\mathscr{G} = R^2 g\, \Delta\rho/\sigma$ and $\Sigma = -R^3 \Omega^2 \Delta\rho/8\sigma$ are, respectively, the gravitational and rotational Bond numbers, $P = p_0 R/\sigma$, and a is the undeformed drop radius. The relative magnitudes of the geostrophic, hydrostatic, and centrifugal pressures at the drop surface are given by

$$\frac{\text{geostrophic}}{\text{centrifugal}} \approx \frac{\mathscr{G}}{\Sigma} \left(\frac{a}{R}\right)^3 \approx \frac{\mathscr{R}_0}{\sqrt{E_k}} \frac{\rho}{\Delta\rho}, \tag{12a}$$

$$\frac{\text{hydrostatic}}{\text{centrifugal}} \approx \frac{\mathscr{G}}{\Sigma} \frac{z}{r^2} \approx \frac{\mathscr{R}_0}{\sqrt{E_k}} \frac{\rho}{\Delta\rho} \left(\frac{R}{a}\right)^3 \frac{z}{r^2}. \tag{12b}$$

In the dynamic regime of interest ($\mathscr{R}_0/\sqrt{E_k} \ll 1$), and in the special case of $\Delta\rho/\rho \approx O(1)$, the geostrophic contribution to the normal stress balance is negligible. The hydrostatic contribution is likewise negligible, except within the necessarily small polar region, $r \lesssim (a/R)^{1/2} (\mathscr{R}_0/\sqrt{E_k})^{1/2}$, wherein the drop shape also depends on the gravitational Bond number $\mathscr{G}$. At leading order, the drop is thus fore–aft symmetric, and the normal stress balance assumes the form describing the shape of a stationary drop in a rotating fluid:

$$P + 4\Sigma r^2 = \nabla_s \cdot \mathbf{n}. \tag{13}$$

The drop shape is determined by a balance between the centrifugal force, which acts to drive the lighter drop fluid along the axis of rotation, and the force owing to interfacial tension and curvature, which tends to maintain the sphericity of the drop. For the case of a buoyant drop, $\Sigma < 0$, the shape is prolate ellipsoidal, with an ellipticity determined by Σ.[12] The solution to (13) has been given by Chandrasekhar,[2] Rosenthal,[4] and Ross.[5,6] The latter author considered the case $\Sigma < 0$ and demonstrated that the prolate ellipsoidal drop shapes are stable to infinitesimal perturbations. For completeness, we outline in the Appendix the method of solution of Chandrasekhar,[2] who considered only the case in which $\Sigma > 0$ and oblate ellipsoidal shapes arise.

Since the inviscid drop has an axisymmetric shape, we may apply the results of Sec. III. The rise speed (9) is set by the drop's equatorial radius R, which is, in turn, implicitly determined by the rotational Bond number Σ. We may thus deduce the drop's velocity as a function of Σ. Figure 2 illustrates the drop's shape and rise speed as a function of the rotational Bond number. The rise speeds have been normalized with respect to that of a spherical inviscid drop. The solid line indicates the rise speed of a deformed inviscid drop

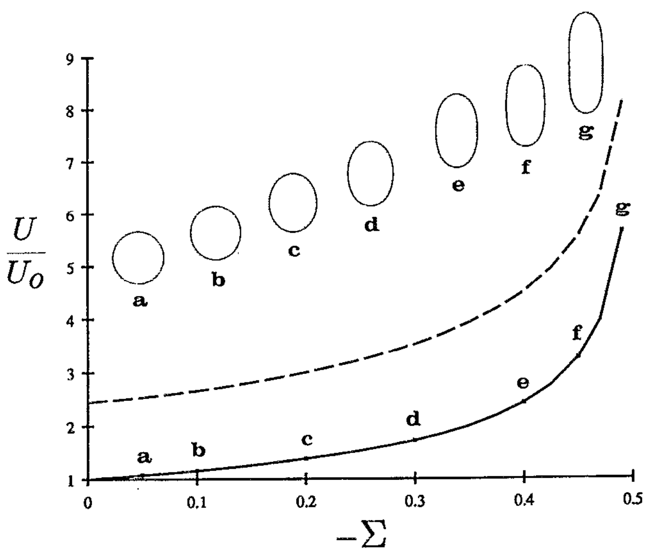

FIG. 2. Rise speed, U_{fr} (solid line), and shape of an inviscid fluid drop bound by surface tension and rising on axis, as a function of the rotational Bond number Σ. The dashed line represents the rise speed, U_{rr}, of an identically shaped rigid body, as given by Moore and Saffman:[2] $U_{rr} = (V/4\pi) \times (\Delta\rho/\rho)\sqrt{\nu/\Omega}\,(g/\Omega)(\int_0^R r^3\, dr/\{1 + [1 + (df/dr)^2]^{1/4}\})^{-1}$, where $z = \pm f(r)$ defines the body's surface. Rise speeds are normalized with respect to that of a spherical inviscid drop, $U_0 = \frac{4}{3}(\Delta\rho/\rho)(g/\Omega a)\sqrt{\nu/\Omega}$, and deformed shapes are scaled such that their volumes correspond to that of the undeformed spherical drop.

and the dashed line that of an identically shaped rigid body. In the limit of large surface tension ($\Sigma \to 0$), the drop is spherical and its rise speed a minimum. As rotational effects become more important, Σ decreases through the range $(0, -\frac{1}{2})$ and the drop becomes progressively more prolate. The rise speed for a drop of fixed volume, which scales with the equatorial radius as R^{-4}, necessarily increases with this progression. In the limit of $\Sigma \to -\frac{1}{2}$, the drop tends toward a cylindrical thread,[6] and the rise speed increases without bound. In this limit, viscous effects are expected to dominate the dynamics ($E_k \gg 1$), so that the geostrophic balance no longer adequately describes the flow.

V. DISCUSSION

Through coupling the analyses of Chandrasekhar[2] and Moore and Saffman,[1] we have deduced the steady shape and rise speed of an inviscid drop rising on axis in a rapidly rotating fluid. In the limit of $(\mathscr{R}_0, E_k, \mathscr{R}_0/\sqrt{E_k}) \ll 1$, the drop is circumscribed by a Taylor column, and its rise induces an azimuthal geostrophic flow in the over- and underlying fluid. While the associated geostrophic pressure field balances the vertical buoyant force on the drop, it has no appreciable effect on the shape, which is determined by a balance between centrifugal and surface tension forces. Our analysis reveals that the equatorial radius of the resulting prolate ellipsoidal drop determines the rise speed. Hence both the steady shape and rise speed are uniquely determined by a single parameter, namely, the rotational Bond number Σ.

We have assumed throughout that the Taylor column extends to the boundaries. Theory predicts that a Taylor column will extend a characteristic distance R/E_k up- and downstream of a body rising slowly on axis through an unbounded fluid.[11] Maxworthy's experimental study[13] revealed columns to be typically an order of magnitude shorter. In order for the Taylor column to span the entire fluid depth, we thus require that the layer depth L be such that $L \lesssim \frac{1}{10} R/E_k$.

Our analysis applies only when geostrophic flow exists in the bulk of the fluid. From Eq. (9) we see that

$$\frac{\mathscr{R}_0}{\sqrt{E_k}} = \frac{4}{3} \frac{g}{a\Omega^2} \left(\frac{a}{R}\right)^4 \frac{\Delta\rho}{\rho}.$$

Thus, in order to achieve the parameter regime $\mathscr{R}_0/\sqrt{E_k} \ll 1$ when $\Delta\rho/\rho \approx O(1)$, we require that $g/a\Omega^2 \ll 1$. For a 1 cm radius air bubble in water, for example, rotation rates on the order of $\Omega \gg 10\ \text{sec}^{-1}$ would be necessary in order to achieve the desired dynamical regime. If low viscosity immiscible fluids of comparable density ($\Delta\rho/\rho \ll 1$) were used, the geostrophic flow regime might be more easily realized experimentally. As is made clear in Eq. (12), however, in this case the hydrostatic and dynamic pressures may no longer be negligible and so may act to destroy the fore–aft symmetry of the drop. An experimental investigation of the results derived herein is currently in progress.

As a final caveat, we note that only in the case considered, namely that of an inviscid drop, is it justifiable to treat the drop surface as a stress-free boundary. For the case of a viscous drop, the internal and external flows couple through viscous boundary layers at the interface. The modification of our results necessitated by the consideration of finite drop viscosity will be the subject of a forthcoming paper.

The theory of buoyancy-driven particle motion in rapidly rotating fluids may be applied in describing the dynamics of the Earth's liquid outer core. According to the dynamo hypothesis of the Earth's magnetic field, convective motions within the outer core are responsible for the sustenance of the geomagnetic field. The Earth's outer core is thought to be composed of a low viscosity, electrically conducting binary fluid comprised of iron and some lighter alloying element. Thermodynamic arguments suggest that core convective motions may be driven to a large extent by chemically rather than thermally induced buoyancy.[14–16] Compositional buoyancy may be generated as iron preferentially freezes out of solution at the inner-core boundary, where a slightly buoyant iron-depleted fluid layer accumulates before going unstable via the Rayleigh–Taylor mechanism and releasing a buoyant "blob" or plume (e.g., Moffatt[17]). The analysis presented here was motivated by an interest in the form of the flows that might be induced by the rise of these buoyant "blobs." A discussion of the application of this and subsequent work to the problem of compositional core convection will be forthcoming.

ACKNOWLEDGMENTS

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APPENDIX: DETAILS OF THE DROP SHAPE CALCULATION

The dimensionless force balance (13) along the axisymmetric drop surface $z = \pm f(r)$ may be expressed as

$$p^* + \frac{1}{2} r^2 = \frac{1}{8} \frac{1}{\Sigma} \nabla_s \cdot \mathbf{n} = -\frac{1}{8} \frac{1}{\Sigma} \frac{1}{r} \frac{d}{dr} \frac{r\phi}{(1+\phi^2)^{1/2}}, \tag{A1}$$

where $\phi = df/dr$ is the local slope of the drop boundary, and $p^* = P/8\Sigma$. Following Chandrasekhar,[2] we outline the few steps that lead to a simple equation describing the drop shape. Integration of (A1) with respect to r yields

$$\frac{p^*}{2} r + \frac{1}{8} r^3 = -\frac{1}{8\Sigma} \frac{\phi}{(1+\phi^2)^{1/2}}. \tag{A2}$$

At the drop equator ($r = 1$), $\phi \to -\infty$, and so

$$\frac{p^*}{2} = \frac{1}{8}\left(\frac{1}{\Sigma} - 1\right).$$

Substitution into (A2) yields

$$\phi/(1+\phi^2)^{1/2} = -r(1 - \Sigma + \Sigma r^2),$$

or, solving for ϕ,

$$\phi = \frac{df}{dr} = -\frac{r(1-\Sigma+\Sigma r^2)}{[1-r^2(1-\Sigma+\Sigma r^2)^2]^{1/2}}.$$

Integration yields

$$z = f(r) = \int_r^1 \frac{r(1-\Sigma+\Sigma r^2)}{[1-r^2(1-\Sigma+\Sigma r^2)^2]^{1/2}}\,dr.$$

Via a series of transformations, this expression can be reduced to an algebraic equation involving elliptic integrals, and so the drop shape $z = f(r)$ may be determined numerically as a function of Σ.

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