Problem 1

a) First we work in the $xy$-plane and then transfer our answer to 3 dimensions. The direction in the $xy$-plane of maximum increase of $z$ is in the direction of $\nabla z = (-4x, -6y) \Rightarrow \nabla z|_{(1,1)} = (-4, -6)$.

⇒ (in the $xy$-plane, the fastest increase is in direction of $(-4, -6)$).

If the projection of her path in the $xy$-plane has tangent vector $ai + bj$ then the tangent vector of her path on the mountain is $ai + bj + (\nabla z \cdot (a, b))k$.

⇒ along the mountain the fastest increase is in direction of $-4i - 6j + 52k$.

b) We need to show that the tangent at each point along the graph of $y = x^{3/2}$ is parallel to $\nabla z$.

Point on graph: $P = (x, x^{3/2})$.

Tangent vector to $y = x^{3/2}$ at $P$ is $v = (1, \frac{3}{2}x^{1/2})$.

$\nabla z|_P = (-4x, -6x^{3/2}) = -4v$. So they are parallel. QED

(The 3d picture at right shows the path along the mountain.

Problem 2

\[ f = x^2 + y^2 + z^2 - 6 \Rightarrow \nabla f = 2x \hat{i} + 2y \hat{j} + 2z \hat{k} \]

⇒ $\vec{N}_i = 2 \hat{i} + 2 \hat{j} + 4 \hat{k}$ is the normal to $f = 0$ at $(1,1,2)$

\[ g = xy^2 - 2 \Rightarrow \nabla g = y^2 \hat{i} + 2xy \hat{j} + xy \hat{k} \]

⇒ $\vec{N}_z = 2 \hat{i} + 2 \hat{j} + \hat{k}$ is the normal to $g = 0$ at $(1,1,2)$

Now $|\vec{N}_i| = \sqrt{2^2 + 2^2 + 4^2} = \sqrt{24} = 2\sqrt{6}$

$|\vec{N}_z| = \sqrt{2^2 + 2^2 + 1^2} = \sqrt{9} = 3$

\[ \cos \theta = \frac{\vec{N}_i \cdot \vec{N}_z}{|\vec{N}_i||\vec{N}_z|} = \frac{(2 \hat{i} + 2 \hat{j} + 4 \hat{k}) \cdot (2 \hat{i} + 2 \hat{j} + \hat{k})}{2\sqrt{6} \cdot 3} = \frac{4 + 4 + 4}{6\sqrt{6}} = \frac{\sqrt{6}}{3} \]
Problem 3.

(2) \( \text{Sketch} \quad \frac{xy^2}{z} = a \quad \text{i.e.} \quad z = \frac{a}{xy} \)

Along \( y = x, \quad z = \frac{a}{x^2} \)
Along \( y = y_0, \quad z = \frac{a}{y_0x} \sim \frac{1}{y_0} \)
Along \( x = x_0, \quad z = \frac{a}{x_0y} \sim \frac{1}{x_0} \)
Also \( z = z_0 \) when \( y = \frac{a}{z_0x} \sim \frac{1}{x} \)

Consider \( W = xy^2 \)

\[ \nabla W = \left< y^2, x^2, xy \right> \quad \text{at} \quad \left( y_{0z}, x, z_0, x_0y \right) \]

Tangent plane has normal \( N = (\nabla W)_x \) and so is given

\[ y_0z_0(x-x_0) + x_0z_0(y-y_0) + x_0y_0(z-z_0) = 0 \]

or since \( x_0y_0z_0 = a \) (since \( P_0 \) on surface):

\[ \frac{y_0z_0}{3} \cdot x + \frac{x_0z_0}{3} \cdot y + \frac{x_0y_0}{3} \cdot z = 3a \]

This intersects the x-axis where \( \left( \frac{z}{y} = 0 \right) \), so at the solution \( y_0z_0x = 3a \)

\[ \cdot x = \frac{3a}{y_0z_0} = 3x_0 \]

By symmetry, it intersects the y-axis and z-axis at \( y_0, z_0 \), respectively.

Volume of resulting tetrahedron = \( \frac{1}{3} \) base \cdot height = \( \frac{1}{3} \cdot 3a \cdot 3a = 9a^2 \).

Problem 4

(a) Line 1: \((0,1,0) + t(0,1,-1) \Rightarrow x = 1, y = t, z = 1 - t.

Line 2: \((0,1,0) + u(1,0,1) \Rightarrow x = u, y = 1, z = u.

b) \( A = (1, t, 1 - t), \quad B = (u, 1, u) \)

\( w(t,u) = |AB|^2 = (u - 1)^2 + (1 - t)^2 + (u + t - 1)^2. \)

\[ \frac{\partial w}{\partial t} = 2(1 - t) + 2(u + t - 1) = 4t + 2u - 4, \]

\[ \frac{\partial w}{\partial u} = 2(u - 1) + 2(u + t - 1) = 4u + 2t - 4. \]

Critical point when \( \frac{\partial w}{\partial t} = \frac{\partial w}{\partial u} = 0 \) \( \Rightarrow \begin{cases} 4t + 2u - 4 = 0, \\ 2t + 4u - 4 = 0. \end{cases} \)

Solving by any method you like \( \Rightarrow u = 2/3, \ t = 2/3. \)

\( A_0 = (1, 2/3, 1/3), \quad B_0 = (2/3, 1, 2/3), \quad |A_0B_0| = 1/\sqrt{3}. \)

(c) \( w_{tt} = 4, \quad w_{uu} = 4, \quad w_{ut} = 2 \Rightarrow D = w_{tt}w_{uu} - w_{ut}^2 = 12 > 0 \Rightarrow \) max or min.

Finally \( w_{tt} > 0 \Rightarrow \) minimum.