18.02A Problem Set 6 – IAP 2010 due Thursday Jan.14, 11:45 in 2-106

Part I (15 points)

Lecture 24. Friday Jan.8  div, grad, curl: physical interpretation
Read: Notes V3, V4  Work: 4E-1ac, 2, 5 (without calculation); 4F-4, 5

Lecture 25. Mon. Jan.11 Extensions of Green’s Theorem
Read: Notes V5, V6  Work: 4G - 1, 4, 5

Read: Notes CV  Work: 3D - 1, 2, 4

Read: Notes I.3, 20.5, 20.6  Work: 5A - 2a,d , 5, 6

Read: 20.7, Notes G

Exam on Lectures 20-27  Friday Jan. 15  12:05-1pm Walker

Part II (35 points)

Directions. Try each problem alone for 15 minutes. If you subsequently collaborate, solutions must be written up independently. It is illegal to consult problem sets from previous semesters. With each problem is the day it can be done.

Problem 1. (Friday. 4pts) Consider the vector field \( \mathbf{F} = 3xy^2 \mathbf{i} - y^3 \mathbf{j} \). Show that the flux over any two curves \( C_1 \) and \( C_2 \) going from the \( x \)- to the \( y \)-axes are the same.

Problem 2. (Friday. 4pts) Consider the force field \( \mathbf{F} = (y + e^x)\mathbf{i} + (x - e^y)\mathbf{j} \). Find the work done on a particle that crosses the first quadrant along the unit circle from \((1,0)\) to \((0,1)\). (Hint: is \( \mathbf{F} \) a gradient field? If so, find and use the potential field.)

Problem 3. (Fri. 5pts)
Let \( C \) be a differentiable curve contained in the portion \( 0 \leq x \leq 2 \) of the first quadrant, starting somewhere on the positive \( y \)-axis and ending somewhere on the line \( x = 2 \). Show that the flux of the vector field \( \mathbf{F} = (8xy - x^4)\mathbf{i} + (2x^3y^2 - 4y^2 - 3x^2)\mathbf{j} \) across \( C \) is always equal to 8. (Hint: apply Green’s Theorem to the region between \( C \) and the \( x \)-axis).

Problem 4. (Mon. 4 pts) Consider \( \mathbf{F} = \left( \frac{-y}{x^2+y^2} \mathbf{i} + \frac{x}{x^2+y^2} \mathbf{j} \right) \).
Show that the flux of \( \mathbf{F} \) across any simple closed curve \( C \) surrounding the origin is zero. (Note that the field is not defined at the origin; thus you should follow the argument presented in the notes for the work done around a simple closed curve by a plane vector field whose curl vanishes, but that is not defined at the origin).

Problem 5. (Mon. 5 pts: 2 + 3)
a) Find the flux of the vector field \( \mathbf{F} = \frac{x}{y} \mathbf{i} + \frac{y}{x} \mathbf{j} \) through the line segment from \((1,-1)\) to \((1,1)\).
b) Find the flux of \( \mathbf{F} \) outwards through any circle centered at \((1, 0)\) of radius \( a \neq 1 \). Consider the cases \( a > 1 \) and \( a < 1 \) separately. Explain your answers. (You may use 4E/5).

Problem 6. (Tues. 5 pts)
Work 3D-7, assuming the planar density is 1, and evaluate the resulting integral in the new coordinate system. Remember that the moment of inertia must be a positive number.

Problem 7 (Wed. 4 pts) Calculate the volume between the surfaces \( x^2 + y^2 + z^2 = 2a^2 \) and \( z = (x^2 + y^2)/a \).

Problem 8 (Wed. 4 pts) Consider a cylinder of unit radius standing vertically, with its centerline along the \( z \)-axis. Calculate the volume of the portion of the cylinder bound by the planes \( z = x \) and \( z = 0 \), and lying in the region \( x > 0 \).