18.02A pset 3, part II solutions, fall 2009

Problem 1

a) \( \nabla C = \left\langle -\frac{2x}{10^4} e^{-(x^2+y^2)/10^4}, -\frac{4y}{10^4} e^{-(x^2+y^2)/10^4} \right\rangle = -\frac{2C(x,y)}{10^4} \langle x, 2y \rangle \).

The radial direction at \((x, y)\) is \( \mathbf{\hat{u}} = \frac{\langle x, y \rangle}{\sqrt{x^2+y^2}} \). \Rightarrow \frac{dC}{ds} \bigg|_{\mathbf{\hat{u}}} = \nabla C \cdot \mathbf{\hat{u}} = -\frac{2C(x,y)}{10^4} \frac{(x^2 + 2y^2)}{\sqrt{x^2+y^2}}.

b) The path must have the same slope as the gradient vector at each point.

That is, \( \frac{dy}{dx} = \frac{2y}{x} \). In 18.01 you learned to separate variables and integrate:

\[
\frac{dy}{y} = \frac{2dx}{x} \Rightarrow \ln y = \ln(x^2) + C \Rightarrow y = Ke^{x^2}.
\]

Using the starting position to determine \( K \), we get the shark’s path is along \( y = \frac{y_0}{x_0^2} x^2 \).

Problem 2

a) The surface is the level surface \( w = z^2 + x^2y^2 + y^3 + x^2 = 4 \).

The normal is \( \nabla w = \langle 2xy^2 + 2x, 2yx^2 + 3y^2, 2z \rangle \). At \((1, 1, 1)\) we get \( \nabla w = \langle 4, 5, 2 \rangle \)

\( \Rightarrow \) the equation of the tangent plane is \( \boxed{4x + 5y + 2z = 11} \).

b) We want to minimize the distance squared \( f(x, y, z) = x^2 + y^2 + z^2 \) subject to the constraint \( 4x + 5y + 2z = 11 \).

Lagrange multipliers gives: \( 2x = 4\lambda, \ 2y = 5\lambda, \ 2z = 2\lambda, \ 4x + 5y + 2z = 11 \).

Substituting for \( x, y, z \) in terms of \( \lambda \) in the constraint gives \( 8\lambda + \frac{25}{2}\lambda + 2\lambda = 11 \Rightarrow \lambda = \frac{22}{45} \)

\( \Rightarrow \) the one critical point is \( \boxed{\frac{11}{45}(4, 5, 2)} \) \( \Rightarrow \) minimum distance = \( \frac{11}{45} \sqrt{45} \).

Problem 3

We want to minimize distance squared \( = x^2 + (y - b)^2 \), subject to the constraint \( y = x^2 \).

The easiest method is to substitute for \( y \).

Thus, distance squared \( = f(x) = x^2 + (x^2 - b)^2 \).

Critical points: \( f'(x) = 2x + 4x(x^2 - b) = 0 \Rightarrow x = 0 \) or \( x^2 = b - 1/2 \).

There are two cases:

(i) \( b > 1/2 \): critical points are \( x = 0 \) and \( x = \pm \sqrt{b - 1/2} \).

Evaluating at the critical points: \( f(0) = b^2, \ f(\pm \sqrt{b - 1/2}) = b - 1/4 \).

Since \( b^2 \geq b - 1/4 \) (because \( b^2 - b + 1/4 = (b - 1/2)^2 \geq 0 \)) the minimum distance in this case is \( \sqrt{b - 1/4} \).

(ii) \( b \leq 1/2 \): the only critical point is \( x = 0 \) \( \Rightarrow \) minimum distance = \( b \).

If you insist on using Lagrange multipliers here it is (the constraint \( x^2 - y = 0 \)).

\( \Rightarrow \ 2x = 2x\lambda, \ 2(y - b) = -\lambda, \ y = x^2 \). As above, there are two cases:

(i) \( \lambda = 1 \) \( \Rightarrow \ y = b - 1/2 = x^2 \): same as case (i) above.

or \( \lambda = 0 \) \( \Rightarrow \ x = 0, \ y = b \): same as case (ii) above.
Problem 4
Before starting we note that if \( \cos a = \cos b \) then either \( b = a \) or \( b = 2\pi - a \).

We start with a circle of radius \( r \) and center \( O \).

Pick three points around the circle with central angles as shown.

Easy trigonometry gives the area of triangle \( OAB = r^2 \cos(\alpha/2) \sin(\alpha/2) = \frac{1}{2} r^2 \sin \alpha \).

Likewise area of \( OBC = \frac{1}{2} r^2 \sin \beta \) and area of \( OCA = \frac{1}{2} r^2 \sin \gamma \).

So, the area of triangle \( ABC = \frac{1}{2} r^2 (\sin \alpha + \sin \beta + \sin \gamma) \), with constraint \( \alpha + \beta + \gamma = 2\pi \).

To find critical points we use Lagrange multipliers:

\[
\frac{1}{2} r^2 \cos \alpha = \lambda, \quad \frac{1}{2} r^2 \cos \beta = \lambda, \quad \frac{1}{2} r^2 \cos \gamma = \lambda, \quad \alpha + \beta + \gamma = 2\pi.
\]

\( \Rightarrow \cos \alpha = \cos \beta = \cos \gamma \).

Our note at the start of the problem implies there are two cases:

(i) All three angles are equal: \( \alpha = \beta = \gamma \).

(ii) Exactly two are the same: say \( \alpha = \beta \) and \( \gamma = 2\pi - \alpha \).

In case (i) the constraint gives \( \alpha = \beta = \gamma = 2\pi/3 \Rightarrow \text{area} = \frac{3r^2}{2} \sin \alpha = \frac{3\sqrt{3}r^2}{4} \).

In case (ii) the constraint gives \( \alpha = \beta = 0, \gamma = 2\pi \Rightarrow \text{area} = 0 \).

Thus the maximum area = \( \frac{3\sqrt{3}r^2}{4} \), for an equilateral triangle.

Problem 5
\( f(x, y) = x^2 - 2xy + 7y^2 \) (objective function).
\( g(x, t) = x^2 + 4y^2 = 1 \) (constraint).

Lagrange: \( \nabla f = \lambda \nabla g \)

\[
\begin{align*}
2x - 2y &= \lambda 2x \\
-2x + 14y &= \lambda 8y \\
x^2 + 4y^2 &= 1
\end{align*}
\]

\( \Rightarrow \begin{cases} 
2x - 2y = \lambda 2x \\
-2x + 14y = \lambda 8y \\
x^2 + 4y^2 = 1
\end{cases} \iff \begin{cases} 
x - y = \lambda x \\
-x + 7y = 4\lambda y \\
x^2 + 4y^2 = 1
\end{cases} \]

There are several methods of solving these equations. (We give only one below.)

They all lead to:

\[
\begin{cases}
(x, y) = (1/\sqrt{5}, -1/\sqrt{5}) \quad \text{or} \quad (-1/\sqrt{5}, 1/\sqrt{5}), \quad f(x, y) = 2, \quad \text{maximum}. \\
(x, y) = (2/\sqrt{5}, 1/2 \sqrt{5}) \quad \text{or} \quad (-2/\sqrt{5}, -1/2 \sqrt{5}), \quad f(x, y) = 3/4, \quad \text{minimum}.
\end{cases}
\]

Solve symmetrically: Take the two equations with \( \lambda \) and multiply to make the left hand sides the same

\( \Rightarrow \begin{cases} 
4xy - 4y^2 = 4\lambda xy \\
x^2 + 7xy = 4\lambda xy \end{cases} \Rightarrow 4xy - 4y^2 = -x^2 + 7xy \Rightarrow 4y^2 = x^2 - 3xy.
\]

The constraint equation can be written as \( 4y^2 = 1 - x^2 \), combining this with the equation just above gives \( x^2 - 3xy = 1 - x^2 \Rightarrow y = \frac{2x^2 - 1}{3x^2} \).

Substitute in the constraint equation \( \Rightarrow x^2 + 4 \left( \frac{2x^2 - 1}{3x^2} \right)^2 = 1 \)

\( \Rightarrow 9x^4 + 16x^4 - 16x^2 + 4 = 9x^2 \Rightarrow 25x^4 - 25x^2 + 4 = 0 \Rightarrow (5x^2 - 4)(5x^2 - 1) = 0 \)

\( \Rightarrow x = \pm 2/\sqrt{5}, \quad \pm 1/\sqrt{5} \).
Now use these $x$ to find $y$ and then evaluate $f(x, y)$ to decide which are minima and which are maxima.

**Problem 6**

a) A little thought shows the rectangle of maximum area must have sides parallel to the axes. So we have area $A = 2xy$, with constraint $x^2 + y^2 = 4$

Lagrange multipliers gives: $2y = 2x\lambda$, $2x = 2y\lambda$, $x^2 + y^2 = 4$.

Solving symmetrically: $\frac{y}{x} = \lambda = \frac{x}{y} \Rightarrow y^2 = x^2$

$\Rightarrow$ (use the constraint) $2x^2 = 4 \Rightarrow x = \sqrt{2} = y$. So, maximum area = 4.

b) Using Lagrange multipliers: $2x = 4\lambda$, $2y = 5\lambda$, $6z = 6\lambda$, $4x + 5y + 6z = 1$.

Substituting for $x$, $y$, $z$ in terms of $\lambda$ in the constraint gives $\frac{53}{2} \lambda = 1 \Rightarrow \lambda = \frac{2}{53}$.

$(x, y, z) = \frac{2}{53}(2, 5/2, 1) = \frac{1}{53}(4, 5, 2). \Rightarrow \text{minimum value} = \frac{1}{53^2}(16 + 25 + 12) = \frac{1}{53}$

**Problem 7**

For starters, $\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle \Rightarrow |\nabla f|^2 = \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2$.

Polar coordinates: $x = r \cos \theta \Rightarrow \frac{\partial x}{\partial r} = \cos \theta$, $\frac{\partial x}{\partial \theta} = -r \sin \theta$.

$y = r \sin \theta \Rightarrow \frac{\partial y}{\partial r} = \sin \theta$, $\frac{\partial y}{\partial \theta} = r \cos \theta$.

Using the chain rule:

$\frac{\partial f}{\partial \theta} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial \theta} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial \theta} = \frac{\partial f}{\partial x} (-r \sin \theta) + \frac{\partial f}{\partial y} r \cos \theta$.

$\frac{\partial f}{\partial r} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial r} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial r} = \frac{\partial f}{\partial x} \cos \theta + \frac{\partial f}{\partial y} \sin \theta$.

$\Rightarrow \left(\frac{1}{r} \frac{\partial f}{\partial \theta}\right)^2 = \left(\frac{\partial f}{\partial x}\right)^2 \sin^2 \theta - 2 \frac{\partial f}{\partial x} \frac{\partial f}{\partial y} \sin \theta \cos \theta + \left(\frac{\partial f}{\partial y}\right)^2 \cos^2 \theta$

$\left(\frac{\partial f}{\partial r}\right)^2 = \left(\frac{\partial f}{\partial x}\right)^2 \cos^2 \theta + 2 \frac{\partial f}{\partial x} \frac{\partial f}{\partial y} \sin \theta \cos \theta + \left(\frac{\partial f}{\partial y}\right)^2 \sin^2 \theta$

Adding these two lines gives $\Rightarrow \left(\frac{1}{r} \frac{\partial f}{\partial \theta}\right)^2 + \left(\frac{\partial f}{\partial r}\right)^2 = \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 = |\nabla f|^2$.

**Problem 8**

Chain rule:

$\frac{dF}{dt} = \frac{\partial F}{\partial x} \frac{dx}{dt} + \frac{\partial F}{\partial y} \frac{dy}{dt} + \frac{\partial F}{\partial z} \frac{dz}{dt}$.

Computing:

$\frac{dx}{dt} = -1$, $\frac{\partial f}{\partial x} = -2xF_0 e^{-x^2+y^2+z^2}$

$\frac{dy}{dt} = 0 \Rightarrow \text{don’t need } \frac{\partial f}{\partial y}$

$\frac{dz}{dt} = -2(1-t)$, $\frac{\partial f}{\partial z} = -2zF_0 e^{-x^2+y^2+z^2}$.

$\Rightarrow \frac{dF}{dt} = -2xF_0 e^{-x^2+y^2+z^2}(-1) - 2zF_0 e^{-x^2+y^2+z^2}(-2)(1-t)$

$= F_0 e^{-(1-t)^2}(-(1-t)^3) (2(1-t) + 4(1-t)^3)$.