18.02A pset 2B, part II solutions, fall 2009

Problem 1

a) $\frac{\partial z}{\partial x} = -4x \Rightarrow \frac{\partial z}{\partial x}(1,1) = -4$. $\frac{\partial z}{\partial y} = -6y \Rightarrow \frac{\partial z}{\partial y}(1,1) = -6$. 

$\Rightarrow$ tangent plane: $(z - 995) = -4(x - 1) - 6(y - 1)$. 

Tangent plane approximation: $\Delta z \approx \frac{\partial z}{\partial x} \Delta x + \frac{\partial z}{\partial y} \Delta y = -4(-.05) - 6(-.05) \Rightarrow \text{approx. change in } z = .5$. 

b) First we work in the $xy$-plane and then transfer our answer to $3$ dimensions. The direction in the $xy$-plane of maximum increase of $z$ is in the direction of $\nabla z = (-4x, -6y)$ \Rightarrow $\nabla z|(1,1) = (-4, -6)$. 

$\Rightarrow$ in the $xy$-plane, the fastest increase is in direction of $(-4, -6)$. 

If the projection of her path in the $xy$-plane has tangent vector $a \mathbf{i} + b \mathbf{j}$ then the tangent vector of her path on the mountain is $a \mathbf{i} + b \mathbf{j} + (\nabla z \cdot \langle a, b \rangle) \mathbf{k}$. 

$\Rightarrow$ along the mountain the fastest increase is in direction of $-4 \mathbf{i} - 6 \mathbf{j} + 52 \mathbf{k}$. 

c) We need to show that the tangent at each point along the graph of $y = x^{3/2}$ is parallel to $\nabla z$. 

Point on graph: \( P = (x, x^{3/2}). \) 

Tangent vector to $y = x^{3/2}$ at $P$ is $\mathbf{v} = (1, \frac{3}{2}x^{1/2})$. 

$\nabla z|_P = (-4x, -6x^{3/2}) = -4x \mathbf{v}$. So they are parallel. QED 

(The picture at right shows the path along the mountain.)

Problem 2

a) If a surface is described by $z = f(x, y)$ then its normal is $\mathbf{N} = (f_x, f_y, -1)$. 

$\Rightarrow$ a normal to the first surface is $\mathbf{N}_1 = (2x, 2y, -1)$. 

$\Rightarrow$ a normal to the second surface is $\mathbf{N}_2 = (1, 1, -1)$. 

At $(1, 1, 2)$: $\mathbf{N}_1 = (2, 2, -1)$ and $\mathbf{N}_2 = (1, 1, -1)$. 

The tangent to the intersection curve is orthogonal to both normals. 

$\Rightarrow$ tangent vector $= \mathbf{N}_1 \times \mathbf{N}_2 = \begin{vmatrix} i & j & k \\ 2 & 2 & -1 \\ 1 & 1 & -1 \end{vmatrix} = -i + j$. 

b. As in part (a): normal to the surfaces are: $\mathbf{N}_1 = (f_x(x_0, y_0), f_y(x_0, y_0), -1)$ and $\mathbf{N}_1 = (g_x(x_0, y_0), g_y(x_0, y_0), -1)$. 

The tangent to the intersection curve is orthogonal to both normals. 

$\Rightarrow$ tangent vector $= \mathbf{N}_1 \times \mathbf{N}_2 = \begin{vmatrix} i & j & k \\ f_x(x_0, y_0) & f_y(x_0, y_0) & -1 \\ g_x(x_0, y_0) & g_y(x_0, y_0) & -1 \end{vmatrix} = (g_y - f_y)i - (g_x - f_x)j + (f_xg_y - f_yg_x)k$. 

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