Problem 1. (30 pts: 10+5+5+5+5)

Consider the points $P = (2, 1, -1)$, $Q = (3, 2, -1)$ and $R = (2, 2, 1)$

a) (10 points) What is the area of triangle $\Delta PQR$?

\[
\overrightarrow{PQ} = \hat{i} + \hat{j} , \quad \overrightarrow{PR} = \hat{j} + 2\hat{k}
\]

\[
\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 0 \\ 0 & 1 & 2 \end{vmatrix} = 2\hat{i} - 2\hat{j} + \hat{k}
\]

\[
|\overrightarrow{PQ} \times \overrightarrow{PR}| = (2^2 + (-2)^2 + 1)^{\frac{1}{2}} = \sqrt{9} = 3
\]

Area of $\Delta PQR = \frac{1}{2} |\overrightarrow{PQ} \times \overrightarrow{PR}| = \frac{3}{2}

b) (5 points) Find a unit normal perpendicular to the plane through $P$, $Q$ and $R$.

\[
\hat{n} = \frac{\overrightarrow{PQ} \times \overrightarrow{PR}}{|\overrightarrow{PQ} \times \overrightarrow{PR}|} = \frac{2\hat{i} - 2\hat{j} + \hat{k}}{3}
\]

(Or $-\hat{n} = -\frac{2\hat{i} + 2\hat{j} - \hat{k}}{3}$)

c) (5 points) Use this unit vector to find the minimum distance from the origin $O = (0, 0, 0)$ to that plane.

\[
\text{Distance} = \hat{n} \cdot \overrightarrow{OP} = \frac{2\hat{i} - 2\hat{j} + \hat{k}}{3} \cdot (2\hat{i} - 2\hat{j} + \hat{k})
\]

\[
= \frac{1}{3} (2^2 - 2^2 + 1) \cdot (2\hat{i} + \hat{j} - \hat{k})
\]

\[
= \frac{1}{3} (4 - 2 - 1) = \frac{1}{3}
\]
d) (5 points) Write the equation of the plane through \( P, Q \) and \( R \) in the form \( ax + by + cz = 1 \).

The eqn of a plane with normal \( (a, b, c) \) passing through a point \( (x_0, y_0, z_0) \) is
\[ ax + by + cz = ax_0 + by_0 + cz_0 \]

Here \( \hat{n} = \frac{1}{3} (2\hat{i} - 2\hat{j} + \hat{k}) \Rightarrow \text{take } (a, b, c) = (2, -2, 1) \)

Choose \( P = (x_0, y_0, z_0) = (2, 1, -1) \)

\[ 2x - 2y + z = 2(2) - 2(1) + 1(-1) = 1 \]

Derived plane: \( 2x - 2y + z = 1 \)

e) (16 points) Compute \( \cos \theta \) where \( \theta \) is the angle between the \( xy \)-plane and the plane defined above.

Normal to \( xy \)-plane: \( \hat{k} \)

Normal to \( PQR \) plane: \( \hat{n} = \frac{1}{3} (2\hat{i} - 2\hat{j} + \hat{k}) \)

The angle between the planes is that between their normals:

\[ \cos \theta = \frac{(\frac{2\hat{i} - 2\hat{j} + \hat{k}}{3}) \cdot \hat{k}}{\frac{1}{3} \sqrt{\left( \frac{2\hat{i} - 2\hat{j} + \hat{k}}{3} \right)^2}} = \frac{\frac{1}{3}}{1} = \frac{1}{3} \]

\( \Rightarrow \theta = \cos^{-1} \left( \frac{1}{3} \right) \)
Problem 2. (15 pts: 5+5+5)

a) (5 points) Calculate the parametric equations of the line parallel to the vector $V = (1, 2, 1)$ and passing through the point $P_0 = (0, 0, 6)$. Use a parameter $t$ to denote the position along the line and so write $x = x(t), y = y(t), z = z(t)$.

\[
\overrightarrow{P_0P} = \vec{V} t = \overrightarrow{OP} - \overrightarrow{OP}_0
\]

\[
(x, y, z) - (x_0, y_0, z_0) = (1, 2, 1) t
\]

\[
(x, y, z) - (0, 0, 6) = (1, 2, 1) t
\]

\[
\Rightarrow x = t, \ y = 2t, \ z = t + 6
\]

b) (5 points) Find an expression for the distance of this line from the origin $D(t)$.

Distance from origin: \[D(t) = \sqrt{x^2 + y^2 + z^2}\]

\[
D^2(t) = t^2 + (2t)^2 + (t + 6)^2
\]

\[
= 6t^2 + 12t + 36
\]

\[
\Rightarrow D(t) = \sqrt{6t^2 + 12t + 36}
\]

c) (5 points) What is the minimum distance of this line from the origin?

Distance is a minimum when

\[
\frac{d}{dt} D^2 = 0 \quad \text{i.e.} \quad 12t + 12 = 0 \Rightarrow t = -1
\]

i.e. at a point \((x, y, z) = (-1, -2, 5)\), where

\[
D = \sqrt{1 + 4 + 25} = \sqrt{30}
\]
Problem 3. (25 pts: 5+5+5+5) Consider the matrix

\[ A = \begin{pmatrix} 0 & 3 & 1 \\ 1 & -1 & 1 \\ 2 & 1 & 0 \end{pmatrix} \]

a) (5 points) Compute \( \det A \).

\[ \det A = -3 \begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} + 1 \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} \]

\[ = -3(-2) + 1(3) \]

\[ = 9 \]

b) (5 points) Find the volume of the parallelepiped formed by the three vectors \( \vec{A} = (0, 3, 1) \), \( \vec{B} = (1, -1, 1) \), \( \vec{C} = (2, 1, 0) \).

\[ \text{Volume} = \det \begin{pmatrix} \vec{A} \\ \vec{B} \\ \vec{C} \end{pmatrix} = \det A = 9 \text{ as above} \]

c) (5 points) Find the entries \( a \) and \( b \) of the inverse matrix:

\[ A^{-1} = \begin{pmatrix} \_ & \_ & \_ \\ \_ & \_ & \_ \\ \_ & \_ & \_ \end{pmatrix} \]

\[ a = \frac{1}{\det A} \begin{vmatrix} -1 & 1 \\ 2 & 0 \end{vmatrix} = \frac{-1}{9}(-2) = \frac{2}{9} \]

\[ b = \frac{1}{\det A} \begin{vmatrix} 0 & 1 \\ 2 & 0 \end{vmatrix} = \frac{1}{9}(-2) = -\frac{2}{9} \]

..... (continued on next page)
Problem 3. (...continued)

d) (5 points) For the system below, say which of $x, y$ or $z$ you can find using only the entries $a$ and $b$ from $A^{-1}$ and find it.

\[
\begin{bmatrix}
-1 \\
3 \\
0
\end{bmatrix}
= A^{-1}
\begin{bmatrix}
1 \\
0 \\
3
\end{bmatrix}
\]

\[
\begin{pmatrix}
x \\
y \\
z
\end{pmatrix}
= \begin{pmatrix}
a & b & * \\
* & * & * \\
* & * & *
\end{pmatrix}
\begin{pmatrix}
1 \\
3 \\
0
\end{pmatrix}
= \begin{pmatrix}
* \\
* \\
*
\end{pmatrix}
\]

so we can obtain $y$:

\[
y = a + 3b = \frac{2}{9} + 3 \left( -\frac{2}{9} \right) = -\frac{4}{9}
\]

e) (5 points) For which values of $c$ does the system below have exactly one solution?

\[
\begin{align*}
3y + cz &= 1 \\
x - y + z &= 3 \\
2x + y &= 0
\end{align*}
\]

\[
A = \begin{pmatrix}
0 & 3 & c \\
1 & -1 & 1 \\
2 & 1 & 0
\end{pmatrix}
\]

There is exactly one solution when $A^-$ exists, that is, when \( \det A \neq 0 \).

\[
\det A = -3 \begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} + c \begin{vmatrix} 1 & 1 \\ 2 & 0 \end{vmatrix} = 6 + 3c
\]

i.e. when $c \neq -2$. 
Problem 4. (30pts: 3+3+5+4+5+5+5)

A soccer ball is kicked at a $45^\circ$ angle to the horizontal with an initial speed $\sqrt{2} V_0$ in the $(i+j)$ direction at a time $t = 0$. If we neglect the influence of air drag, the trajectory of the ball's center is given by:

$$r(t) = V_0 t \hat{i} + (V_0 t - \frac{1}{2} g t^2) \hat{j},$$

where $g$ is the gravitational acceleration.

a) (3 points) the velocity vector $\mathbf{V}$,

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = V_0 \hat{i} + (V_0 - gt) \hat{j}$$

b) (3 points) the acceleration vector $\mathbf{a}$,

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{d^2\mathbf{r}}{dt^2} = -g \hat{j}$$

c) (5 points) the speed $\frac{ds}{dt}$,

$$\frac{ds}{dt} = |\mathbf{v}| = \left[ V_0^2 + (V_0 - gt)^2 \right]^{\frac{1}{2}}$$

d) (4 points) the unit tangent vector $\mathbf{T}$.

$$\mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{1}{\sqrt{V_0^2 + (V_0 - gt)^2}} \left[ V_0 \hat{i} + (V_0 - gt) \hat{j} \right]$$

... (continued on next page)
Problem 4. (...continued)

e) (5 points) Find the curvature $\kappa$.

\[
\kappa = \frac{x'' y' - y'' x'}{(x'^2 + y'^2)^{3/2}}
\]

where \[x(t) = V_0 t, \quad x'' = 0 \]
\[y(t) = V_0 t - \frac{1}{2} g t^2 \]
\[y''(t) = V_0 - gt, \quad y''' = -g \]

\[\Rightarrow \kappa = -\frac{V_0 g}{\left[V_0^2 + (V_0 - gt)^2\right]^{3/2}}\]

f) (5 points) Write an integral expression for the total distance traced out between take-off ($t = 0$) and landing ($t = 2V_0/g$). (You need not evaluate the integral).

Arc length \[S = \int_{t=0}^{2V_0/g} \frac{ds}{dt} dt = \int_{t=0}^{2V_0/g} \sqrt{V_0^2 + (V_0 - gt)^2} dt\]

g) (5 points) If the ball has radius $a$ and is struck with backspin so that it rotates counterclockwise $M$ times per unit time, use vector methods to write the parametric equations $x = x(t), y = y(t)$ for a point on the ball, specifically, the point $P$ initially at $(a, 0)$.

\[\vec{OP} = \vec{OA} + \vec{AP}\]

\[\vec{OA} = V_0 t \hat{i} + \left(V_0 t - \frac{1}{2} gt^2\right) \hat{j}\]
\[\vec{AP} = a \cos 2\pi M t \hat{i} + a \sin 2\pi M t \hat{j}\]

\[\Rightarrow x(t) = V_0 t + a \cos 2\pi M t\]
\[y(t) = V_0 t - \frac{1}{2} gt^2 + a \sin 2\pi M t\]