A Crystal Definition for Symplectic Multiple Dirichlet Series

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1 Introduction

This paper presents a definition for a family of Weyl group multiple Dirichlet series (henceforth "MDS") of Cartan type C using a combinatorial model for crystal bases due to Berenstein-Zelevinsky [2] and Littelmann [12]. Recall that a Weyl group MDS is a Dirichlet series in several complex variables which (at least conjecturally) possesses analytic continuation to a meromorphic function and satisfies functional equations whose action on the complex space is isomorphic to the given Weyl group. In [1], we presented a definition for such a series in terms of a basis for highest weight representations of $Sp(2r, \mathbb{C})$ – Type C Gelfand-Tsetlin patterns – and proved that the series satisfied the conjectured analytic properties in a number of special cases. Here we recast that definition in the language of crystal bases and find that the resulting MDS, whose form appears as an unmotivated miracle in the language of Gelfand-Tsetlin patterns, is more naturally defined in this new language.

The family of MDS is indexed by a positive integer r, an odd positive integer n, and an r-tuple of non-zero algebraic integers $\mathbf{m} = (m_1, \ldots, m_r)$ from a ring described precisely in Section 3. In [1], we further conjectured (and proved for $n = 1$) that this series matches the $(m_1, \ldots, m_r)$-th Whittaker coefficient of a minimal parabolic, metaplectic Eisenstein series on an n-fold cover of $SO(2r + 1)$ over a suitable choice of global field. It is known that the definition of MDS we present fails to have the conjectured analytic properties if n is even, reflecting the essential interplay between n and root lengths in our definition (see, for example, Section 3.6). An alternate definition for Weyl group MDS attached to any root system (with completely general choice of r, n, and $\mathbf{m}$) was given by Chinta and Gunnells [8], who proved they possess analytic continuation and functional equations. Our definition of MDS for type C is conjecturally equal to theirs, and this has been verified in a large number of special cases.

The remainder of the paper is outlined as follows. In Sections 2 and 3, we recall the model for the crystal basis from [12] and basic facts about Weyl group MDS for any root system Φ. In Section 4, we define the MDS coefficients in terms of crystal bases, and explain their relation to our earlier definition in [1]. (It is instructive to compare this definition with that of [7].) Section 5 demonstrates how, for any fixed choice of data determining a single MDS of type C, one may prove that the resulting series satisfies the conjectured functional equations. Similar techniques would be applicable to Weyl group MDS for any root system. As demonstrated in [5], by Bochner's theorem in several complex variables, the existence of such functional equations then leads to a proof of the desired meromorphic continuation to the entire complex space $\mathbb{C}^r$.

The method for proving functional equations for a given Dirichlet series relies on reduction to the rank one case, whose analytic properties were demonstrated by Kubota [11]. Similar techniques were employed in [5] and [6], where the definition of the Dirichlet series was much simpler having assumed that the defining datum n is sufficiently large. Our methods indicate that the same should be true for arbitrary choice of n and arbitrary root system, leading to several potential applications. First, if one is interested in mean-value estimates for coefficients appearing in a given Weyl group MDS, this method provides the necessary analytic information to apply standard Tauberian techniques. More generally, one may take residues of the Weyl group MDS to obtain a further class of Dirichlet series with analytic continuation. In computing these residues, it is often useful to first express them in terms of rank one Kubota Dirichlet series given by our method. (For a similar example in type A, see [4].)

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2 Littelmann's polytope basis for crystals

Given a semisimple algebraic group G of rank r and a simple G-module V_μ of highest weight μ, we may associate a crystal graph X_μ to V_μ. That is, there exists a corresponding simple module for the quantum group $U_q(\mathrm{Lie}(G))$ having the associated crystal graph structure. Roughly speaking, the crystal graph encodes data from the representation V_μ, and should be regarded as a kind of "enhanced character" for the representation; for an introduction to crystal graphs, see [10] or [9]. For now, we merely recall that the vertices of X_μ are in bijection with a basis of weight vectors for the highest weight representation and the r "colored" edges of X_μ correspond to simple roots $\alpha_1, \ldots, \alpha_r$ of G. Two vertices b_1, b_2 are connected by a (directed) edge from b_1 to b_2 of color i if the Kashiwara lowering operator f_{α_i} takes b_1 to b_2. If the vertex b has no outgoing edge of color i, we set $f_{\alpha_i}(b) = 0$.

Littelmann gives a combinatorial model for the crystal graph as follows. Fix a reduced decomposition of the long element w_0 of the Weyl group of G into simple reflections σ_i:

$$w_0 = \sigma_{i_1}\sigma_{i_2}\cdots\sigma_{i_N}.$$

Given an element b (i.e. vertex) of the crystal X_μ, let t_1 be the maximal integer such that $b_1 := f_{\alpha_{i_1}}^{t_1}(b) \neq 0$. Similarly, let t_2 be the maximal integer such that $b_2 := f_{\alpha_{i_2}}^{t_2}(b_1) \neq 0$. Continuing in this fashion in order of the simple reflections appearing in w_0, we obtain a string of non-negative integers $(t_1(b), \ldots, t_N(b))$. We often suppress this dependence on b, and simply write $(t_1, \ldots, t_N)$. Note that by well-known properties of the crystal graph, we are guaranteed that for any reduced decomposition of the long element and an arbitrary element b of the crystal, the path $f_{\alpha_{i_N}}^{t_N}\cdots f_{\alpha_{i_1}}^{t_1}(b)$ through the crystal always terminates at b_{low}, corresponding to the lowest weight vector of the crystal graph X_μ.

Littelmann proves that, for any fixed reduced decomposition, the set of all sequences $(t_1, \ldots, t_N)$ as we vary over all vertices of all highest weight crystals V_μ associated to G fill out the integer lattice points of a cone in $\mathbb{R}^N$. The inequalities describing the boundary of this cone depend on the choice of reduced decomposition. For a particular "nice" subset of the set of all reduced decompositions, Littelmann shows that the cone is defined by a rather simple set of inequalities. (A precise definition of "nice" and numerous examples may be found in [12] and we will only make use of one such example.) For any fixed highest weight μ, the set of all sequences $(t_1, \ldots, t_N)$ for the crystal X_μ are the integer lattice points of a polytope in $\mathbb{R}^N$. The boundary of the polytope consists of the hyperplanes defined by the cone inequalities independent of μ, together with additional hyperplanes dictated by the choice of μ.

We now describe this geometry in the special case of $Sp_{2r}(\mathbb{C})$. We fix an enumeration of simple roots chosen so that α_1 is the unique long simple root and α_i and α_{i+1} correspond to adjacent nodes in the Dynkin diagram. This example is dealt with explicitly in Section 6 of [12] with the following "nice decomposition" of the long element of the associated Weyl group:

$$w_0 = \sigma_1(\sigma_2\sigma_1\sigma_2)(\ldots)(\sigma_{r-1}\ldots\sigma_1\ldots\sigma_{r-1})(\sigma_r\sigma_{r-1}\ldots\sigma_1\ldots\sigma_{r-1}\sigma_r). \tag{1}$$

With $N = r^2$, let $\mathbf{t} = (t_1, t_2, \ldots, t_N)$ be the string generated by traversing the crystal graph from a given weight b to the highest weight μ as described above. An example in rank 2 is given in Figure 1.

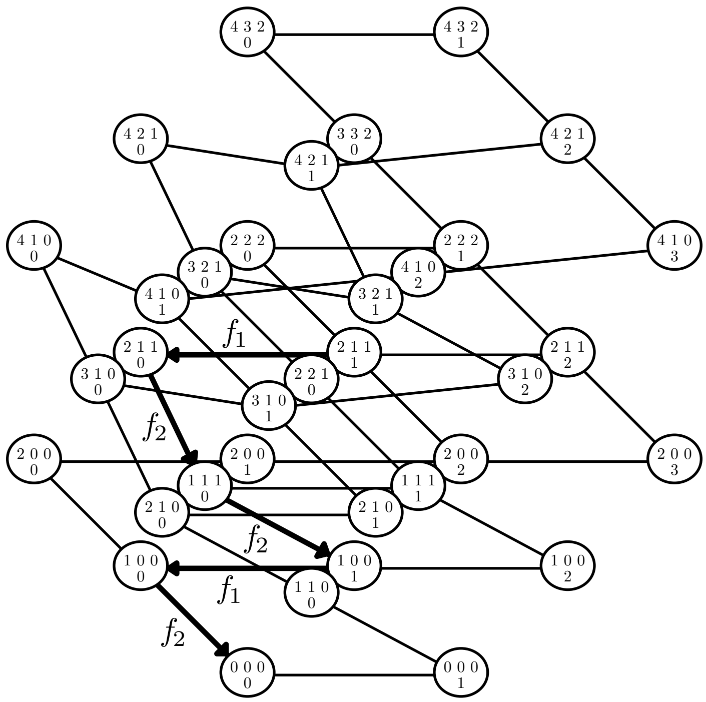


Figure 1: The crystal graph for a highest weight representation V_λ of $Sp(4)$ with $\lambda = \epsilon_1 + 2\epsilon_2$ (ϵ_i : fundamental dominant weights). The vertices of the graph have been labeled with their corresponding sequence (t_1, t_2, t_3, t_4) obtained by traversing the graph by maximal paths in the Kashiwara lowering operators in the respective order (f_1, f_2, f_1, f_2). This order is determined by the decomposition of $w_0 = \sigma_1\sigma_2\sigma_1\sigma_2$. For each vertex, t_1 is centered in the bottom row, and the top row is (t_2, t_3, t_4) read left to right. The highlighted path demonstrates this for the vertex labeled $(1, 2, 1, 1)$. The picture has been drawn so that vertices that touch represent basis vectors in the same weight space.

In order to describe the cone inequalities for $Sp(2r, \mathbb{C})$ with w_0 as in (1), it is convenient to place the sequence $\mathbf{t} = (t_1, t_2, \ldots, t_N)$ in a triangular array. Following Littelmann [12], construct a triangle Δ consisting of r centered rows of boxes, with $2(r+1-i)-1$ entries in the row i, starting from the top. To any vector $\mathbf{t} \in \mathbb{R}^{r^2}$, let $\Delta(\mathbf{t})$ denote the filled triangle whose entries are the coordinates of $\mathbf{t}$, with the boxes filled from the bottom row to the top row, and from left to right. For notational ease, we reindex the entries of Δ using standard matrix notation; let $c_{i,j}$ denote the

j-th entry in the i-th row of Δ, with $i \leq j \leq 2r - i$. Also, for convenience in the discussion below, we will write $\overline{c}_{i,j} := c_{i,2r-j}$ for $i \leq j \leq r$. Thus when $r = 3$, we are considering triangles of the form

$c_{1,1}$	$c_{1,2}$	$c_{1,3}$	$\overline{c}_{1,2}$	$\overline{c}_{1,1}$
	$c_{2,2}$	$c_{2,3}$	$\overline{c}_{2,2}$	
		$c_{3,3}$		

so that, for example,

$$\mathbf{t} = (2,2,1,1,5,3,2,2,1) \quad \mapsto \quad \Delta(\mathbf{t}) = \begin{array}{|c|c|c|c|c|} \hline 5 & 3 & 2 & 2 & 1 \\ \hline & 2 & 1 & 1 & \\ \cline{2-4} & & 2 & & \\ \cline{3-3} \end{array} \ .$$

Given this notation, we may now state the cone inequalities.

Proposition 1 (Littelmann, [12], Theorem 6.1) *For $G = Sp(2r, \mathbb{C})$ and w_0 as in (1), the corresponding cone of all sequences $\mathbf{t}$ is given by the set of all triangles $\Delta(\mathbf{t})$ with non-negative entries $\{c_{i,j}\}$ that are weakly decreasing in rows.*

Recall that for any fundamental dominant weight μ, the set of all paths $\mathbf{t}$ ranging over all vertices of the crystal X_μ are the integer lattice points of a polytope $\mathcal{C}_\mu$ in $\mathbb{R}^N$. We now describe the remaining hyperplane inequalities which define this polytope.

Proposition 2 (Littelmann, [12] Corollary 6.1) *Let $G = Sp_{2r}(\mathbb{C})$ and let w_0 be as in (1). Write $\mu = \mu_1\epsilon_1 + \cdots + \mu_r\epsilon_r$, with ϵ_i the fundamental weights. Then $\mathcal{C}_\mu$ is the convex polytope of all triangles $\Delta(c_{i,j})$ such that the entries in the rows are non-negative and weakly increasing, and satisfy the following upper-bound inequalities for all $1 \leq i \leq r$ and $1 \leq j \leq r-1$:*

$$\overline{c}_{i,j} \leq \mu_{r-j+1} + s(\overline{c}_{i,j-1}) - 2s(c_{i-1,j}) + s(c_{i-1,j+1}), \tag{2}$$

$$c_{i,j} \leq \mu_{r-j+1} + s(\overline{c}_{i,j-1}) - 2s(\overline{c}_{i,j}) + s(c_{i,j+1}), \tag{3}$$

$$\text{and} \quad c_{i,r} \leq \mu_1 + s(\overline{c}_{i,r-1}) - s(c_{i-1,r}). \tag{4}$$

In the above, we have set

$$s(\overline{c}_{i,j}) := \overline{c}_{i,j} + \sum_{k=1}^{i-1}(c_{k,j} + \overline{c}_{k,j}), \quad s(c_{i,j}) := \sum_{k=1}^{i}(c_{k,j} + \overline{c}_{k,j}), \quad s(c_{i,m}) := \sum_{k=1}^{i} 2c_{k,m}.$$

We will call these triangular arrays "Berenstein-Zelevinsky-Littelmann patterns" (or "BZL-patterns" for short). The set of patterns corresponding to all vertices in a highest weight crystal X_μ will be referred to as $BZL(\mu)$.

3 Definition of the multiple Dirichlet series

In this section, we give the general shape of a Weyl group MDS, beginning with the rank one case. In particular, we reduce the determination of the higher rank Dirichlet series to it's prime-power supported coefficients, which will be given in the next section as a generating function over BZL-patterns.

3.1 Algebraic preliminaries

Given a fixed positive integer n, let F be a number field containing the $2n^{\text{th}}$ roots of unity, and let S be a finite set of places containing all ramified places over $\mathbb{Q}$, all archimedean places, and enough additional places so that the ring of S-integers $\mathcal{O}_S$ is a principal ideal domain. Recall that

$$\mathcal{O}_S = \{a \in F \ | \ a \in \mathcal{O}_v \ \forall v \notin S\},$$

and can be embedded diagonally in $F_S = \prod_{v \in S} F_v$. There exists a pairing

$$(\cdot,\cdot)_S \, : \, F_S^\times \times F_S^\times \longrightarrow \mu_n \text{ defined by } (a,b)_S = \prod_{v\in S} (a,b)_v,$$

where the $(a,b)_v$ are local Hilbert symbols associated to n and v.

To any $a \in \mathcal{O}_S$ and any ideal $\mathfrak{b} \in \mathcal{O}_S$, we may associate the nth power residue symbol $\left(\frac{a}{\mathfrak{b}}\right)_n$ as follows. For prime ideals $\mathfrak{p}$, the expression $\left(\frac{a}{\mathfrak{p}}\right)_n$ is the unique n^{th} root of unity satisfying the congruence

$$\left(\frac{a}{\mathfrak{p}}\right)_n \equiv a^{(N(\mathfrak{p})-1)/n} \ (\text{mod } \mathfrak{p}).$$

We then extend the symbol to arbitrary ideals $\mathfrak{b}$ by multiplicativity, with the convention that the symbol is 0 whenever a and $\mathfrak{b}$ are not relatively prime. Since $\mathcal{O}_S$ is a principal ideal domain by assumption, we will write

$$\left(\frac{a}{b}\right)_n = \left(\frac{a}{\mathfrak{b}}\right)_n \quad \text{for } \mathfrak{b} = b\mathcal{O}_S$$

and often drop the subscript n on the symbol when the power is clear from context.

Then if a, b are coprime integers in $\mathcal{O}_S$, we have the nth power reciprocity law (cf. [13], Thm. 6.8.3)

$$\left(\frac{a}{b}\right) = (b,a)_S \left(\frac{b}{a}\right). \tag{5}$$

Lastly, for a positive integer t and $a, c \in \mathcal{O}_S$ with $c \neq 0$, we define the Gauss sum $g_t(a,c)$ as follows. First, choose a non-trivial additive character ψ of F_S trivial on the $\mathcal{O}_S$ integers (cf. [3] for details). Then the n^{th}-power Gauss sum is given by

$$g_t(a,c) = \sum_{d \bmod c} \left(\frac{d}{c}\right)_n^t \psi\left(\frac{ad}{c}\right), \tag{6}$$

where we have suppressed the dependence on n in the notation on the left.

3.2 Kubota's rank one Dirichlet series

We now present Kubota's Dirichlet series arising from the Fourier coefficient of an Eisenstein series on an n-fold cover of $SL(2, F_S)$. It is the prototypical Weyl group MDS and many of the general definitions of Section 3.4 can be understood as natural extensions of those in the rank one case. Moreover, we will make repeated use of the functional equation for the Kubota Dirichlet series when we demonstrate the functional equations for higher rank MDS by reduction to rank one in Section 6.

A subgroup $\Omega \subset F_S^\times$ is said to be *isotropic* if $(a,b)_S = 1$ for all $a, b \in \Omega$. In particular, $\Omega = \mathcal{O}_S (F_S^\times)^n$ is isotropic (where $(F_S^\times)^n$ denotes the n^{th} powers in $F_S^\times$). Let $\mathcal{M}_t(\Omega)$ be the space of functions $\Psi : F_S^\times \longrightarrow \mathbb{C}$ that satisfy the transformation property

$$\Psi(\epsilon c) = (c, \epsilon)_S^{-t} \Psi(c) \quad \text{for any } \epsilon \in \Omega, c \in F_S^\times. \tag{7}$$

For $\Psi \in \mathcal{M}_t(\Omega)$, consider the "Kubota Dirichlet series"

$$\mathcal{D}_t(s, \Psi, a) = \sum_{0 \neq c \in \mathcal{O}_s / \mathcal{O}_s^\times} \frac{g_t(a,c)\Psi(c)}{|c|^{2s}}. \tag{8}$$

Here $|c|$ is the order of $\mathcal{O}_S / c\mathcal{O}_S$, $g_t(a,c)$ is as in (6) and the term $g_t(a,c)\Psi(c)|c|^{-2s}$ is independent of the choice of representative c, modulo S-units. Standard estimates for Gauss sums show that the series is convergent if $\Re(s) > \frac{3}{4}$. To state a precise functional equation, we require the Gamma factor

$$\mathbf{G}_n(s) = (2\pi)^{-2(n-1)s} n^{2ns} \prod_{j=1}^{n-2} \Gamma\left(2s - 1 + \frac{j}{n}\right). \tag{9}$$

In view of the multiplication formula for the Gamma function, we may also write

$$\mathbf{G}_n(s) = (2\pi)^{-(n-1)(2s-1)} \frac{\Gamma(n(2s-1))}{\Gamma(2s-1)}.$$

Let

$$\mathcal{D}_t^*(s, \Psi, a) = \mathbf{G}_m(s)^{[F:\mathbb{Q}]/2} \zeta_F(2ms - m + 1) \mathcal{D}_t(s, \Psi, a), \tag{10}$$

where $m = n/\gcd(n,t)$, $\frac{1}{2}[F:\mathbb{Q}]$ is the number of archimedean places of the totally complex field F, and ζ_F is the Dedekind zeta function of F.

If $v \in S_{\mathrm{fin}}$, the non-archimedean places of S, let q_v denote the cardinality of the residue class field $\mathcal{O}_v/\mathcal{P}_v$, where $\mathcal{O}_v$ is the local ring in F_v and $\mathcal{P}_v$ is its prime ideal. By an *S-Dirichlet polynomial* we mean a polynomial in q_v^{-s} as v runs through the finitely many places of S_{fin}. If $\Psi \in \mathcal{M}_t(\Omega)$ and $\eta \in F_S^\times$, denote

$$\widetilde{\Psi}_\eta(c) = (\eta, c)_S \, \Psi(c^{-1}\eta^{-1}). \tag{11}$$

Then we have the following result (Theorem 1 in [6]), which follows from the work of Brubaker and Bump [3].

Theorem 1 *Let $\Psi \in \mathcal{M}_t(\Omega)$ and $a \in \mathcal{O}_S$. Let $m = n/\gcd(n,t)$. Then $\mathcal{D}_t^*(s, \Psi, a)$ has meromorphic continuation to all s, analytic except possibly at $s = \frac{1}{2} \pm \frac{1}{2m}$, where it may have simple poles. There exist S-Dirichlet polynomials $P_\eta^t(s)$ depending only on the image of η in $F_S^\times/(F_S^\times)^n$ such that*

$$\mathcal{D}_t^*(s, \Psi, a) = |a|^{1-2s} \sum_{\eta \in F_S^\times/(F_S^\times)^n} P_{a\eta}^t(s) \mathcal{D}_t^*(1-s, \widetilde{\Psi}_\eta, a). \tag{12}$$

This result, based on ideas of Kubota [11], relies on the theory of Eisenstein series. The case $t = 1$ is handled in [3]; the general case follows as discussed in the proof of Proposition 5.2 of [5]. Notably, the factor $|a|^{1-2s}$ is independent of the value of t.

3.3 Root systems

Before proceeding to the definition of higher-rank MDS, which uses the language of the associated root system, we first fix notation and recall a few basic results.

Let Φ be a reduced root system contained in a real vector space V of dimension r. The dual vector space $V^\vee$ contains a root system $\Phi^\vee$ in bijection with Φ, where the bijection switches long and short roots. If we write the dual pairing

$$V \times V^\vee \longrightarrow \mathbb{R} : \quad (x, y) \mapsto B(x, y), \tag{13}$$

then $B(\alpha, \alpha^\vee) = 2$. Moreover, the simple reflection $\sigma_\alpha : V \to V$ corresponding to α is given by

$$\sigma_\alpha(x) = x - B(x, \alpha^\vee)\alpha.$$

In particular σ_α preserves Φ. Similarly we define $\sigma_{\alpha^\vee} : V^\vee \to V^\vee$ by $\sigma_{\alpha^\vee}(x) = x - B(\alpha, x)\alpha^\vee$ with $\sigma_{\alpha^\vee}(\Phi^\vee) = \Phi^\vee$.

Without loss of generality, we may take Φ to be irreducible (i.e., there do not exist orthogonal subspaces Φ_1, Φ_2 with $\Phi_1 \cup \Phi_2 = \Phi$). Then set $\langle \cdot, \cdot \rangle$ to be the Euclidean inner product on V and $||\alpha|| = \sqrt{\langle \alpha, \alpha \rangle}$ the Euclidean norm, normalized so that $2\langle \alpha, \beta \rangle$ and $||\alpha||^2$ are integral for all $\alpha, \beta \in \Phi$. With this notation, we may alternately write

$$\sigma_\alpha(\beta) = \beta - \frac{2\langle \beta, \alpha \rangle}{\langle \alpha, \alpha \rangle}\alpha \quad \text{for any } \alpha, \beta \in \Phi. \tag{14}$$

We partition Φ into positive roots Φ^+ and negative roots Φ^- and let $\Delta = \{\alpha_1, \ldots, \alpha_r\} \subset \Phi^+$ denote the subset of simple positive roots. Let ϵ_i for $i = 1, \ldots, r$ denote the fundamental dominant weights satisfying

$$\frac{2\langle \epsilon_i, \alpha_j \rangle}{\langle \alpha_j, \alpha_j \rangle} = \delta_{ij}, \quad \delta_{ij} : \text{Kronecker delta}. \tag{15}$$

Any dominant weight λ is expressible as a non-negative linear combination of the ϵ_i, and a distinguished role in the theory is played by the Weyl vector ρ, defined by

$$\rho = \frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha = \sum_{i=1}^{r} \epsilon_i. \tag{16}$$

3.4 The form of higher rank multiple Dirichlet series

We now begin explicitly defining the multiple Dirichlet series, retaining our previous notation. By analogy with the rank 1 definition in (7), given an isotropic subgroup Ω, let $\mathcal{M}(\Omega^r)$ be the space of functions $\Psi : (F_S^\times)^r \longrightarrow \mathbb{C}$ that satisfy the transformation property

$$\Psi(\epsilon \mathbf{c}) = \left(\prod_{i=1}^{r} (\epsilon_i, c_i)_S^{||\alpha_i||^2} \prod_{i<j} (\epsilon_i, c_j)_S^{2\langle \alpha_i, \alpha_j \rangle} \right) \Psi(\mathbf{c}) \tag{17}$$

for all $\epsilon = (\epsilon_1, \ldots, \epsilon_r) \in \Omega^r$ and all $\mathbf{c} = (c_1, \ldots, c_r) \in (F_S^\times)^r$.

Given a reduced root system Φ of fixed rank r, an integer $n \geq 1$, $\mathbf{m} \in \mathcal{O}_S^r$, and $\Psi \in \mathcal{M}(\Omega^r)$, then we define a multiple Dirichlet series as follows. It is a function of r complex variables $\mathbf{s} = (s_1, \ldots, s_r) \in \mathbb{C}^r$ of the form

$$Z_\Psi(\mathbf{s};\mathbf{m}) := Z_\Psi(s_1, \ldots, s_r; m_1, \ldots, m_r) = \sum_{\mathbf{c}=(c_1,\ldots,c_r)\in(\mathcal{O}_S/\mathcal{O}_S^\times)^r} \frac{H^{(n)}(\mathbf{c};\mathbf{m})\Psi(\mathbf{c})}{|c_1|^{2s_1}\cdots|c_r|^{2s_r}}. \quad (18)$$

The function $H^{(n)}(\mathbf{c};\mathbf{m})$ carries the main arithmetic content. In general it is not a multiplicative function, but rather a "twisted multiplicative" function. That is, for S-integer vectors $\mathbf{c}, \mathbf{c}' \in (\mathcal{O}_S/\mathcal{O}_S^\times)^r$ with $\gcd(c_1\cdots c_r, c_1'\cdots c_r') = 1$,

$$H^{(n)}(c_1c_1', \ldots, c_rc_r'; \mathbf{m}) = \mu(\mathbf{c}, \mathbf{c}')H^{(n)}(\mathbf{c};\mathbf{m})H^{(n)}(\mathbf{c}';\mathbf{m}) \quad (19)$$

where $\mu(\mathbf{c}, \mathbf{c}')$ is an n^{th} root of unity depending on $\mathbf{c}, \mathbf{c}'$. It is given precisely by

$$\mu(\mathbf{c}, \mathbf{c}') = \prod_{i=1}^{r} \left(\frac{c_i}{c_i'}\right)_n^{||\alpha_i||^2} \left(\frac{c_i'}{c_i}\right)_n^{||\alpha_i||^2} \prod_{i<j} \left(\frac{c_i}{c_j'}\right)_n^{2\langle\alpha_i,\alpha_j\rangle} \left(\frac{c_i'}{c_j}\right)_n^{2\langle\alpha_i,\alpha_j\rangle} \quad (20)$$

where $\left(\frac{\cdot}{\cdot}\right)_n$ is the n^{th} power residue symbol defined in Section 3.1. Note that in the special case $\Phi = A_1$, the twisted multiplicativity in (19) and (20) agrees with the usual identity for Gauss sums appearing in the numerator for the rank one case given in (8).

The transformation property of functions in $\mathcal{M}(\Omega^r)$ in (17) above is, in part, motivated by the identity

$$H^{(n)}(\epsilon\mathbf{c};\mathbf{m})\Psi(\epsilon\mathbf{c}) = H^{(n)}(\mathbf{c};\mathbf{m})\Psi(\mathbf{c}) \quad \text{for all } \epsilon \in \mathcal{O}_S^r, \mathbf{c}, \mathbf{m} \in (F_S^\times)^r.$$

This can be verified using the n^{th} power reciprocity law from Section 3.1.

The function $H^{(n)}(\mathbf{c};\mathbf{m})$ also exhibits a twisted multiplicativity in $\mathbf{m}$. Given any $\mathbf{m}, \mathbf{m}', \mathbf{c} \in \mathcal{O}_S^r$ with $\gcd(m_1'\cdots m_r', c_1\cdots c_r) = 1$, we let

$$H^{(n)}(\mathbf{c}; m_1m_1', \ldots, m_rm_r') = \prod_{i=1}^{r} \left(\frac{m_i'}{c_i}\right)_n^{-||\alpha_i||^2} H^{(n)}(\mathbf{c};\mathbf{m}). \quad (21)$$

The definitions in (19) and (21) imply that it is enough to specify the coefficients $H^{(n)}(p^{k_1}, \ldots, p^{k_r}; p^{l_1}, \cdots, p^{l_r})$ for any fixed prime p with $l_i = \text{ord}_p(m_i)$ in order to completely determine $H^{(n)}(\mathbf{c};\mathbf{m})$ for any pair of S-integer vectors $\mathbf{m}$ and $\mathbf{c}$. These prime-power coefficients are described in terms of data from highest-weight representations associated to $(l_1, \cdots, l_r)$ and will be given precisely in Section 4.

3.5 Weyl group actions

In order to precisely state a functional equation for the Weyl group multiple Dirichlet series defined in (18), we require an action of the Weyl group W of Φ on the complex parameters $(s_1, \dots, s_r)$. This arises from the linear action of W, realized as the group generated by the simple reflections $\sigma_{\alpha^\vee}$, on $V^\vee$. From the perspective of Dirichlet series, it is more natural to consider this action shifted by $\rho^\vee$, half the sum of the positive co-roots. Then each $w \in W$ induces a transformation $V_{\mathbb{C}}^\vee = V^\vee \otimes \mathbb{C} \to V_{\mathbb{C}}^\vee$ (still denoted by w) if we require that

$$B(w\alpha, w(\mathbf{s}) - \frac{1}{2}\rho^\vee) = B(\alpha, \mathbf{s} - \frac{1}{2}\rho^\vee).$$

We introduce coordinates on $V_{\mathbb{C}}^\vee$ using simple roots $\Delta = \{\alpha_1, \dots, \alpha_r\}$ as follows. Define an isomorphism $V_{\mathbb{C}}^\vee \to \mathbb{C}^r$ by

$$\mathbf{s} \mapsto (s_1, s_2, \dots, s_r) \quad s_i = B(\alpha_i, \mathbf{s}). \tag{22}$$

This action allows us to identify $V_{\mathbb{C}}^\vee$ with $\mathbb{C}^r$, and so the complex variables s_i that appear in the definition of the multiple Dirichlet series may be regarded as coordinates in either space. It is convenient to describe this action more explicitly in terms of the s_i and it suffices to consider simple reflections which generate W. Using the action of the simple reflection σ_{α_i} on the root system Φ given in (14) in conjunction with (22) above gives the following:

Proposition 3 *The action of σ_{α_i} on $\mathbf{s} = (s_1, \dots, s_r)$ defined implicitly in (22) is given by*

$$s_j \mapsto s_j - \frac{2\langle \alpha_j, \alpha_i \rangle}{\langle \alpha_i, \alpha_i \rangle}\left(s_i - \frac{1}{2}\right) \quad j = 1, \dots, r. \tag{23}$$

In particular, $\sigma_{\alpha_i} : s_i \mapsto 1 - s_i$. For convenience, we will write σ_i for σ_{α_i}.

3.6 Normalizing factors and functional equations

The multiple Dirichlet series must also be normalized using Gamma and zeta factors in order to state precise functional equations. Let

$$n(\alpha) = \frac{n}{\gcd(n, ||\alpha||^2)}, \quad \alpha \in \Phi^+.$$

For example, if $\Phi = C_r$ and we normalize short roots to have length 1, this implies that $n(\alpha) = n$ unless α is a long root and n even (in which case $n(\alpha) = n/2$). By analogy with the zeta factor appearing in (10), for any $\alpha \in \Phi^+$, let

$$\zeta_\alpha(\mathbf{s}) = \zeta\left(1 + 2n(\alpha)B(\alpha, \mathbf{s} - \frac{1}{2}\rho^\vee)\right)$$

where ζ is the Dedekind zeta function attached to the number field F. Further, for $\mathbf{G}_n(s)$ as in (9), we may define

$$\mathbf{G}_\alpha(\mathbf{s}) = \mathbf{G}_{n(\alpha)}\left(\frac{1}{2} + B(\alpha, \mathbf{s} - \frac{1}{2}\rho^\vee)\right). \tag{24}$$

Then for any $\mathbf{m} \in \mathcal{O}_S^r$, the normalized multiple Dirichlet series is given by

$$Z_\Psi^*(\mathbf{s}; \mathbf{m}) = \left[\prod_{\alpha \in \Phi^+} \mathbf{G}_\alpha(\mathbf{s})\zeta_\alpha(\mathbf{s})\right] Z_\Psi(\mathbf{s}, \mathbf{m}). \tag{25}$$

For any fixed n, $\mathbf{m}$ and root system Φ, we seek to exhibit a definition for $H^{(n)}(\mathbf{c}; \mathbf{m})$ (or equivalently, given twisted multiplicativity, a definition of H at prime-power coefficients) such that $Z_\Psi^*(\mathbf{s}; \mathbf{m})$ satisfies functional equations of the form:

$$Z_\Psi^*(\mathbf{s}; \mathbf{m}) = |m_i|^{1-2s_i} Z_{\sigma_i\Psi}^*(\sigma_i\mathbf{s}; \mathbf{m}) \tag{26}$$

for all simple reflections $\sigma_i \in W$. Here, $\sigma_i\mathbf{s}$ is as in (23) and the function $\sigma_i\Psi$, which essentially keeps track of the rather complicated scattering matrix in this functional equation, is defined as in (37) of [6]. As noted in Section 7 of [6], given functional equations of this type, one can obtain analytic continuation to a meromorphic function of $\mathbb{C}^r$ with an explicit description of polar hyperplanes.

4 Definition of the prime-power coefficients

In this section, we use crystal graphs to give a definition for the p-power coefficients $H^{(n)}(p^{\mathbf{k}}; p^{\mathbf{l}})$ in a multiple Dirichlet series for the root system C_r with n odd. More precisely, the p-power coeffcients will be given as weighted sums over the BZL-patterns defined in Section 2 that have weight corresponding to $\mathbf{k}$. Given a fixed r-tuple of integers $\mathbf{l} = (l_1, \ldots, l_r)$, let

$$\lambda = \sum_{i=1}^{r} l_i\epsilon_i, \tag{27}$$

where ϵ_i are fundamental dominant weights. The contributions to $H^{(n)}(p^{\mathbf{k}};p^{\mathbf{l}})$ are parametrized by basis vectors of the highest weight representation with highest weight $\lambda+\ \rho$, where ρ is the Weyl vector for C_r defined in (16). We use the set of BZL-patterns $BZL(\lambda+\rho)$ as our combinatorial model for these basis vectors.

The contributions to each $H^{(n)}(p^{\mathbf{k}};p^{\mathbf{l}})$ come from a single weight space corresponding to $\mathbf{k}=(k_1,\ldots,k_r)$ in the highest weight representation $\lambda+\rho$ corresponding to $\mathbf{l}$. Given a BZL-pattern $\Delta=\Delta(c_{i,j})$, define the vector

$$k(\Delta)\ =\ (k_1(\Delta),k_2(\Delta),\ldots,k_r(\Delta))$$

with

$$k_1(\Delta)=\sum_{i=1}^{r}c_{i,r},\quad\text{and}\quad k_j(\Delta)=\sum_{i=1}^{r+1-j}(c_{i,r+1-j}+\overline{c}_{i,r+1-j}),\quad\text{for } 1<j\leq r. \tag{28}$$

We define

$$H^{(n)}(p^{\mathbf{k}};p^{\mathbf{l}})=H^{(n)}(p^{k_1},\ldots,p^{k_r};p^{l_1},\ldots,p^{l_r})=\sum_{\substack{\Delta\in BZL(\lambda+\rho)\\ k(\Delta)=(k_1,\ldots,k_r)}}G(\Delta) \tag{29}$$

where $G(\Delta)$ is a weighting function to be defined presently.

To this end, we will apply certain *decoration rules* to the BZL-patterns. These decorations will consist of *boxes* and *circles* around the individual entries of the pattern, applied according to the following rules:

1. The entry $c_{i,j}$ is circled if $c_{i,j}=c_{i,j+1}$. We understand the entries outside the triangular array to be zeroes, so the right-most entry in a row will be circled if it equals 0.

2. The entry $c_{i,j}$ is boxed if equality holds in the upper-bound inequality of Proposition 2 having $c_{i,j}$ as the lone term on the left-hand side.

We illustrate these rules in the following rank 3 example. Let $(l_1,l_2,l_3)=(0,1,1)$. Then there are 9 upper bound inequalities for the polytope $\mathcal{C}_{\lambda+\rho}$. We state them for the five top row elements $c_{1,j}$, leaving the rest to the reader:

$$\overline{c}_{1,1}\leq 2,\quad \overline{c}_{1,2}\leq 2+\overline{c}_{1,1}\quad c_{1,3}\leq 1+\overline{c}_{1,2}$$
$$c_{1,2}\leq 2+\overline{c}_{1,1}-2\overline{c}_{1,2}+c_{1,3},\quad c_{1,1}\leq 2-2\overline{c}_{1,1}+c_{1,2}+\overline{c}_{1,2}.$$

We may now decorate any pattern occurring in $BZL(\lambda+\rho)$. For example, the following BZL-pattern (with decorations) occurs in this set:

$$\begin{array}{ccccc} \boxed{5} & \boxed{3} & \textcircled{2} & 2 & 1 \\ & 2 & \textcircled{1} & 1 & \\ & & 2 & & \end{array} . \tag{30}$$

To each entry $c_{i,j}$ in a *decorated* $\Delta(\mathbf{c})$, we associate the complex-valued function

$$\gamma(c_{i,j}) = \begin{cases} q^{c_{i,j}} & \text{if } c_{i,j} \text{ is circled (but not boxed),} \\ g_1(p^{c_{i,j}-1}, p^{c_{i,j}}) & \text{if } c_{i,j} \text{ is boxed (but not circled), and } j \neq r, \\ g_2(p^{c_{i,j}-1}, p^{c_{i,j}}) & \text{if } c_{i,j} \text{ is boxed (but not circled), and } j = r, \\ \phi(p^{c_{i,j}}) & \text{if } c_{i,j} \text{ is neither boxed nor circled,} \\ 0 & \text{if } c_{i,j} \text{ is both boxed and circled,} \end{cases} \tag{31}$$

where $g_t(p^\alpha, p^\beta)$ is an n^{th}-power Gauss sum as in (6), $\phi(p^a)$ denotes Euler's totient function for $\mathcal{O}_S/p^a\mathcal{O}_S$, and $q = |\mathcal{O}_S/p\mathcal{O}_S|$. Then at last, we may define the weighting function appearing in (29) by

$$G(\Delta) = \prod_{\substack{1 \leq i \leq r, \\ i \leq j \leq 2r-1}} \gamma(c_{i,j}). \tag{32}$$

For instance, in (30), we find that

$$\gamma(c_{1,1}) = g_1(p^4, p^5), \quad \gamma(c_{1,3}) = q^2, \quad \text{ and } \gamma(\bar{c}_{1,2}) = \phi(p^2).$$

Computing the remaining $\gamma(c_{i,j})$'s for Δ in (30), we have

$$G(\Delta) = \{g_1(p^4, p^5) g_1(p^2, p^3) q^2 \phi(p^2) \phi(p)\} \cdot \{\phi(p^2) q \phi(p)\} \cdot \phi(p^2).$$

Note that the definition implies that some BZL-patterns Δ will have $G(\Delta) = 0$. For instance, in rank 2 with $l_1 = 3$ and $l_2 = 4$, the decorated pattern

$$\Delta = \begin{array}{ccc} \boxed{\textcircled{5}} & \boxed{5} & \textcircled{5} \\ & 3 & \end{array}$$

occurs, and has $G(\Delta) = 0$.

The definition of $G(\Delta)$ in (32) completes the definition of the prime-power coefficients $H^{(n)}(p^{\mathbf{k}}; p^{\mathbf{l}})$ in (29). According to the twisted multiplicativity given in Section 3.4, this completely determines the coefficients of the multiple Dirichlet series $Z_\Psi(\mathbf{s}; \mathbf{m})$ defined in (18).

5 Equality of the GT and BZL descriptions

In Section 3 of [1], we gave an alternate definition for the p-power coefficients using Gelfand-Tsetlin patterns (henceforth "GT-patterns") as our combinatorial model for the highest weight representation. In this section, we will demonstrate that the two definitions for p-power coefficients $H^{(n)}(p^{\mathbf{k}};p^{\mathbf{l}})$ in terms of GT-patterns and BZL-patterns are indeed the same.

A GT-pattern P associated to $Sp(2r,\mathbb{C})$ has the form

$$P = \begin{array}{cccccccccc} a_{0,1} & & a_{0,2} & & \cdots & & a_{0,r} & \\ & b_{1,1} & & b_{1,2} & \cdots & b_{1,r-1} & & b_{1,r} \\ & & a_{1,2} & & \cdots & & a_{1,r} & \\ & & & \ddots & & \ddots & & \vdots \\ & & & & & & a_{r-1,r} & \\ & & & & & & & b_{r,r} \end{array} \tag{33}$$

where the $a_{i,j}, b_{i,j}$ are non-negative integers and the rows of the pattern interleave. That is, for all $a_{i,j}, b_{i,j}$ in the pattern P above,

$$\min(a_{i-1,j}, a_{i,j}) \geq b_{i,j} \geq \max(a_{i-1,j+1}, a_{i,j+1})$$

and

$$\min(b_{i+1,j-1}, b_{i,j-1}) \geq a_{i,j} \geq \max(b_{i+1,j}, b_{i,j}).$$

A careful summary of patterns of this type arising from branching rules for classical groups can be found in [14] building on the work of [16].

Let $\lambda + \rho = (l_1+1)\epsilon_1 + \cdots + (l_r+1)\epsilon_r$ and set

$$(L_r, \cdots, L_1) := (l_1 + l_2 + \cdots + l_r + r, \ldots, l_1 + l_2 + 2, l_1 + 1). \tag{34}$$

Then the set of all GT-patterns with top row $(a_{0,1}, \ldots, a_{0,r}) = (L_r, \ldots, L_1)$ forms a basis for the highest weight representation with highest weight $\lambda + \rho$. We refer to this set of patterns as $GT(\lambda + \rho)$.

Proposition 4 (Littelmann, [12] Corollary 6.2) *The following equations induce a bijection of sets φ between $GT(\lambda+\rho)$ and $BZL(\lambda+\rho)$:*

$$\begin{aligned} \overline{c}_{i,j} &= \sum_{m=1}^{j} \left(a_{i-1,m} - b_{i,m}\right), \quad \textit{for } i \leq j \leq r, \\ \textit{and} \quad c_{i,j} &= \overline{c}_{i,r} + \sum_{m=1}^{r-j} \left(a_{i,r+1-m} - b_{i,r+1-m}\right), \quad \textit{for } i < j \leq r-1. \end{aligned} \tag{35}$$

Remark 1 The map given in Corollary 2 of Section 6 in [12] is actually the inverse of the map defined by (35). (Note that there are several typographical errors in the presentation of the map in [12].) In that same section, Littelmann gives an example illustrating this correspondence, in the case of rank 3. This example is given below, with the corrected first entry in the second row.

$$
\begin{array}{cccccc}
9 & & 5 & & 1 & \\
 & 6 & & 5 & & 0 \\
 & & 5 & & 3 & \\
 & & & 5 & & 2 \\
 & & & & 3 & \\
 & & & & & 1
\end{array}
\qquad \longleftrightarrow \qquad
\begin{array}{ccccc}
\boxed{7} & \boxed{7} & \boxed{4} & \boxed{3} & \boxed{3} \\
 & \boxed{2} & \boxed{1} & \boxed{0} & \\
 & & \boxed{2} & &
\end{array}
$$

Using the bijection of the previous proposition, we may now compare the two definitions for prime-power coefficients of the multiple Dirichlet series.

Proposition 5 *Given a fundamental dominant weight λ, let G_{GT} be the function defined on Gelfand-Tsetlin patterns P in $GT(\lambda+\rho)$ in Definition 3 of [1]. Let $G(\Delta)$ be the function defined on BZL patterns in (32). Then, with φ the bijection of Proposition 4,*

$$G_{GT}((P)) = G(\varphi(P)).$$

Proof It suffices to check that the cases defining the function on GT-patterns match those for BZL-patterns. Indeed, one must check that "maximal" and "minimal" entries in GT-patterns correspond to boxing and circling, respectively, in BZL-patterns. This is a simple consequence of the bijection in Proposition 4, and we leave the case analysis to the reader. □

6 Functional equations by reduction to rank one

In this section, we provide evidence toward global functional equations for the multiple Dirichlet series $Z_\Psi(\mathbf{s};\mathbf{m})$ through a series of computations in a particular rank 2 example. We will demonstrate that these multiple Dirichlet series are, in some sense, built from combinations of rank 1 Kubota Dirichlet series and thus inherit their functional equations. Similar techniques to those presented here would apply for arbitrary rank.

Recall from (23) that, in rank 2, we expect functional equations corresponding to the simple reflections

$$\sigma_1 : (s_1, s_2) \mapsto (1 - s_1, s_1 + s_2 - 1/2) \text{ and } \sigma_2 : (s_1, s_2) \mapsto (s_1 + 2s_2 - 1, 1 - s_2), \quad (36)$$

which generate a group acting on $(s_1, s_2) \in \mathbb{C}^2$ isomorphic to the Weyl group of C_2, the dihedral group of order 8.

With notations as before, let $n = 3$, and $\mathbf{m} = (p^2, p^1)$ for some fixed $\mathcal{O}_S$ prime p. Then we will illustrate how our definition of the coefficients $H^{(3)}(\mathbf{c}; p^2, p)$ leads to a multiple Dirichlet series $Z_\Psi(\mathbf{s}; p^2, p)$ satisfying the functional equations

$$Z_\Psi(s_1, s_2; p^2, p) \to |p^2|^{1-2s_1} Z_{\sigma_1 \Psi}(1 - s_1, s_1 + s_2 - 1/2; p^2, p) \tag{37}$$

and

$$Z_\Psi(s_1, s_2; p^2, p) \to |p|^{1-2s_2} Z_{\sigma_2 \Psi}(s_1 + 2s_2 - 1, 1 - s_2; p^2, p) \tag{38}$$

corresponding to the above simple reflections according to (26).

Our strategy is quite simple. To demonstrate the functional equation corresponding to σ_1, write

$$Z_\Psi(s_1, s_2; p^2, p) = \sum_{c_2 \in \mathcal{O}_S / \mathcal{O}_S^\times} |c_2|^{-2s_2} \sum_{c_1 \in \mathcal{O}_S / \mathcal{O}_S^\times} \frac{H^{(3)}(c_1, c_2; p^2, p) \Psi(\mathbf{c})}{|c_1|^{2s_1}} \tag{39}$$

and attempt to realize the inner sum, for any fixed c_2, in terms of rank 1 Kubota Dirichlet series whose one-variable functional equations are all compatible with the global functional equation in (37). Similar methods apply for the other simple reflection. One difficulty with this approach is that our definitions for $H^{(n)}(\mathbf{c}; \mathbf{m})$ up to this point have been "local" – that is, we have only provided explicit definitions for the prime power supported coefficients. Of course, our requirement that the $H^{(n)}(\mathbf{c}; \mathbf{m})$ satisfy twisted multiplicativity then uniquely defines the coefficients for any r-tuple of integers $\mathbf{c}$, but there are many complications in attempting to patch together the prime-power supported pieces to reconstruct a global series.

This strategy was precisely carried out in [5] and [6] for any root system Φ provided n is sufficiently large. Such values of n are referred to as *stable* (see [6] for the precise statement). Indeed, global objects were reconstructed from the prime-power supported contributions by meticulously checking that all Hilbert symbols and n^{th} power residue symbols combine neatly into Kubota Dirichlet series with the required twisted multiplicativity. Our purpose here is not to get bogged down in these complications, but rather to show how global functional equations can be anticipated simply by considering the prime-power supported coefficients. According to [6], the example with $\mathbf{m} = (p^2, p)$ will have a simple description only if $n \geq 7$, hence when $n = 3$, the results of [6] do not apply. Nevertheless, as we will explain, our method of reduction to the rank 1 case is still viable.

6.1 Analysis of $H^{(3)}(c_1, c_2; p^2, p)$ with prime-power support

The nature of $H^{(3)}(c_1, c_2; p^2, p)$ with c_1, c_2 powers of a fixed prime depends critically on whether that prime is p, the fixed prime occurring in $\mathbf{m} = (p^2, p)$, or a distinct prime $\ell \neq p$. The prime-power supported coefficients $H^{(3)}(\ell^{k_1}, \ell^{k_2}; p^2, p)$ at primes $\ell \neq p$ have identical support (k_1, k_2) for any such prime ℓ (as the support depends only on $\text{ord}_\ell(m_1)$ and $\text{ord}_\ell(m_2)$) and a uniform description as products of Gauss sums in terms of ℓ. The (k_1, k_2) coordinates of this support are depicted in Figure 2 – the result of the affine linear transformation of the weights in the corresponding highest weight representation ρ. The vertex in the bottom left corner is placed at $(k_1, k_2) = (0, 0)$. At each of the vertices in the interior, the number shown indicates the number of BZL-patterns associated with that vertex, that is, the multiplicity in the associated weight space. These counts include both singular and non-singular patterns, though singular patterns give no contribution to the multiple Dirichlet series for any n. Support on the boundary is indicated by black dots, each with a unique corresponding BZL-pattern.

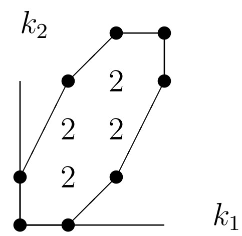


Figure 2: Support (k_1, k_2) for $H^{(3)}(\ell^{k_1}, \ell^{k_2}; p^2, p)$ (with indicated multiplicities of contributing BZL patterns Δ having $k(\Delta) = (k_1, k_2)$).

For $n = 3$ each of the 8 patterns Δ (4 singular, 4 non-singular) in the interior of the polygon of support have $G(\Delta) = 0$, so the only non-zero contributions come from the 8 boundary vertices. Note that these are just the "stable" vertices, which have $G(\Delta)$ non-zero for all n.

The coefficients $H^{(3)}(p^{k_1}, p^{k_2}; p^2, p^1)$ are much more interesting. Recall these coefficients are parametrized by BZL-patterns with the coordinates of $\lambda + \rho$ given by $(L_2, L_1) = (5, 3)$, as in (34). The supporting vertices (k_1, k_2) for the p-part are shown below in Figure 3. On the support's boundary, stable vertices are indicated by filled circles and unstable vertices are indicated by open circles, all with multiplicity one.

Again, the choice of $n = 3$ will make $G(\Delta) = 0$ for many of the patterns Δ occurring at these support vertices. Roughly speaking, the non-zero support for any fixed n forms an $n \times n$ regular lattice beginning at the origin. However, this

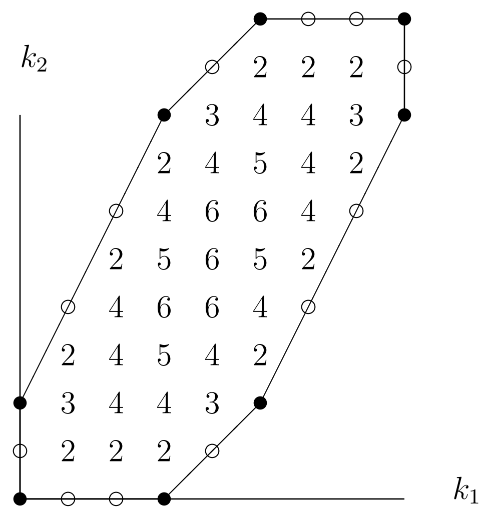


Figure 3: Support (k_1, k_2) for $H^{(3)}(p^{k_1}, p^{k_2}; p^2, p)$ (with indicated multiplicities of contributing BZL patterns Δ).

lattice becomes somewhat distorted by the boundary of the polygon, particularly the location of the stable vertices. In fact, our choice of $(L_2, L_1) = (5, 3)$ in this example is so small that this phenomenon is essentially obscured.

6.2 Three specific examples

Returning to the discussion of functional equations, we will first demonstrate a functional equation corresponding to the simple reflection σ_1 taking $s_1 \mapsto 1 - s_1$. Recall our strategy is to show that for any choice of c_2, we may write the inner sum in (39) in terms of Kubota Dirichlet series. For example, let $c_2 = p^8$. By twisted multiplicativity, we see that $H^{(3)}(c_1, p^8; p^2, p)$ will be 0 unless $\mathrm{ord}_\ell(c_1) \leq 1$ for all primes $\ell \neq p$ (as evident from Figure 2, since we seek ℓ-power terms with support $k_2 = 0$). More interestingly, using Figure 3, we see that p-power terms with $k_2 = 8$ must have $3 \leq \mathrm{ord}_p(c_1) \leq 8$. Let's examine the p-power coefficients more closely.

6.2.1 The functional equation σ_1 with $k_2 = 8$

As seen in Figure 3, $H^{(3)}(p^{k_1}, p^{k_2}; p^2, p)$ with $k_2 = 8$ has support at 6 lattice points $(k_1, 8)$ with a total of 16 BZL-patterns. Having chosen $n = 3$ (so that all Gauss sums appearing are formed with a cubic residue symbol), one checks that only five of these 16 BZL-patterns have non-zero Gauss sum products associated to them. These are listed in the table below.

Δ	$k(\Delta)$	$G(\Delta)$	$G(\Delta)$ for $n=3$
$\begin{array}{ccc} \boxed{8} & \boxed{3} & \textcircled{0} \\ & \textcircled{0} & \end{array}$	$(3,8)$	$g_2(p^2,p^3)\, g_1(p^7,p^8)$	$-\lvert p\rvert^2 g_1(p^7,p^8)$
$\begin{array}{ccc} 6 & \boxed{5} & \boxed{2} \\ & \textcircled{0} & \end{array}$	$(5,8)$	$g_1(p^1,p^2)\, g_2(p^4,p^5)\, g_1(p^7,p^6)$	$\lvert p\rvert^6 \phi(p^6)$
$\begin{array}{ccc} \boxed{8} & \boxed{3} & \textcircled{0} \\ & 3 & \end{array}$	$(6,8)$	$g_2(p^2,p^3)\, g_1(p^7,p^8)\, g_2(p^4,p^3)$	$-\lvert p\rvert^2 g_1(p^7,p^8)\phi(p^3)$
$\begin{array}{ccc} 6 & \boxed{5} & \boxed{2} \\ & \boxed{1} & \end{array}$	$(6,8)$	$g_1(p^1,p^2)\, g_2(p^4,p^5)\, g_1(p^7,p^6)\, g_2(1,p)$	$\lvert p\rvert^6 \phi(p^6) g_2(1,p)$
$\begin{array}{ccc} \boxed{8} & \boxed{3} & \textcircled{0} \\ & \boxed{5} & \end{array}$	$(8,8)$	$g_2(p^2,p^3)\, g_1(p^7,p^8)\, g_2(p^4,p^5)$	$-\lvert p\rvert^2 g_1(p^7,p^8) g_2(p^4,p^5)$

We have computed the final column in the table from the third column, using the following three elementary properties of n^{th}-order Gauss sums at prime powers, which can be proved easily from the definition in (6):

1. If $a \geq b$, then $g_t(p^a, p^b) = \begin{cases} \phi(p^b) & n \mid tb, \\ 0 & n \nmid tb. \end{cases}$
2. For any integers a and t, $g_t(p^{a-1}, p^a) = |p|^{a-1} g_{at}(1, p)$.
3. For any integer t, $g_t(1,p)\, g_{n-t}(1,p) = |p|$.

For notational convenience, let the inner sum in (39) be denoted

$$F(s_1; c_2) = \sum_{c_1 \in \mathcal{O}_S/\mathcal{O}_S^\times} \frac{H^{(3)}(c_1, c_2; p^2, p)\Psi(\mathbf{c})}{|c_1|^{2s_1}}. \tag{40}$$

Fix $c_2 = p^8$ and let

$$F^{(p)}(s_1;p^8) = \sum_{k_1} \frac{H^{(3)}(p^{k_1},p^8;p^2,p)\Psi(p^{k_1},p^8)}{|p|^{2k_1 s_1}}. \tag{41}$$

From the table above, this sum is supported at $k_1 = 3, 5, 6$ and 8, so that $F^{(p)}(s_1;p^8)$ equals

$$\frac{-|p|^2\, g_1(p^7,p^8)\Psi(p^3,p^8)}{|p|^{6s_1}} \left[1 + \frac{g_2(p^4,p^3)}{|p|^{6s_1}}\frac{\Psi(p^6,p^8)}{\Psi(p^3,p^8)} + \frac{g_2(p^4,p^5)}{|p|^{10s_1}}\frac{\Psi(p^8,p^8)}{\Psi(p^3,p^8)}\right] + \frac{|p|^6\phi(p^6)\Psi(p^5,p^8)}{p^{10s_1}}\left[1 + \frac{g_2(1,p)}{|p|^{2s_1}}\cdot\frac{\Psi(p^6,p^8)}{\Psi(p^5,p^8)}\right] \tag{42}$$

Ignoring complications from the Ψ function, both bracketed sums may be expressed as the p-part of a Kubota Dirichlet series in s_1. Indeed, letting $\mathcal{D}_2^{(p)}$ denote the prime-power supported coefficients of the Kubota Dirichlet series $\mathcal{D}_2$ in (8), then

$$\mathcal{D}_2^{(p)}(s_1,\Psi',p^4) = \left[1 + \frac{g_2(p^4,p^3)}{|p|^{6s_1}}\frac{\Psi(p^6,p^8)}{\Psi(p^3,p^8)} + \frac{g_2(p^4,p^5)}{|p|^{10s_1}}\frac{\Psi(p^8,p^8)}{\Psi(p^3,p^8)}\right]$$

for some appropriately defined $\Psi' \in \mathcal{M}_2(\Omega)$, as $\mathcal{D}_2^{(p)}(s_1,\Psi',p^4)$ contains $g_2(p^4,p^{k_1})$ in the numerator, which is non-zero only if $k_1 = 0, 3$ or 5 when $n = 3$. Similarly,

$$\mathcal{D}_2^{(p)}(s_1,\Psi'',1) = \left[1 + \frac{g_2(1,p)}{|p|^{2s_1}}\cdot\frac{\Psi(p^6,p^8)}{\Psi(p^5,p^8)}\right]$$

for an appropriately defined $\Psi'' \in \mathcal{M}_2(\Omega)$. Thus, according to (42), we may express $F^{(p)}(s_1)$ as the sum of p-parts of Kubota Dirichlet series multiplied by Dirichlet monomials. The reader interested in checking all details regarding the Ψ function should refer to Section 5 of [5]; our notation for the one-variable Ψ' or Ψ'' in $M_2(\Omega)$ derived from $\Psi(c_1,c_2)$ is called Ψ^{c_1,c_2} in Lemma 5.3 of [5].

In order to reconstruct the global object $F(s_1;c_2)$ with $c_2 = p^8$, we now turn to the analysis at primes $\ell \neq p$. Since $\mathrm{ord}_\ell(c_2) = 0$, then we can reconstruct $F(s_1;p^8)$ from the twisted multiplicativity in (19) and (21) together with knowledge of terms of the form $H^{(3)}(\ell^{k_1},1;p^2,p)$. Then define

$$F^{(\ell)}(s_1;1) = \sum_{k_1} \frac{H^{(3)}(\ell^{k_1},1;p^2,p)\Psi(\ell^{k_1},p^8)}{|\ell|^{2k_1 s_1}}$$

for all primes $\ell \neq p$. Using twisted multiplicativity in (21),

$$\begin{aligned} F^{(\ell)}(s_1;1) &= \sum_{k_1} \left(\frac{p^2}{\ell^{k_1}}\right)_3^{-2} \left(\frac{p}{1}\right)_3^{-1} \frac{H^{(3)}(\ell^{k_1},1;1,1)\Psi(\ell^{k_1},p^8)}{|\ell|^{2k_1 s_1}} \\ &= \Psi(1,p^8) + \left(\frac{p^2}{\ell}\right)_3^{-2} H^{(3)}(\ell^1,1;1,1)\Psi(\ell^1,p^8)|\ell|^{-2s_1} \\ &= \Psi(1,p^8)\left[1 + \left(\frac{p^2}{\ell}\right)_3^{-2} \frac{g_2(1,\ell)}{|\ell|^{2s_1}} \frac{\Psi(\ell^1,p^8)}{\Psi(1,p^8)}\right]. \end{aligned}$$

To summarize, we have found that

$$F^{(p)}(s_1;p^8) = \frac{-|p|^2\, g_1(p^7,p^8)\Psi(p^3,p^8)}{|p|^{6s_1}}\mathcal{D}_2^{(p)}(s_1,\Psi',p^4) + \frac{|p|^6\phi(p^6)\Psi(p^5,p^8)}{|p|^{10s_1}}\mathcal{D}_2^{(p)}(s_1,\Psi'',1)$$

and

$$F^{(\ell)}(s_1;1) = \Psi(1,p^8)\left[1 + \left(\frac{p^2}{\ell}\right)_3^{-2} \frac{g_2(1,\ell)}{|\ell|^{2s_1}} \frac{\Psi(\ell^1,p^8)}{\Psi(1,p^8)}\right], \text{ for all primes } \ell \neq p.$$

Now using twisted multiplicativity, we can reconstruct $F(s_1;p^8)$. We claim that

$$F(s_1;p^8) = \frac{-|p|^2\, g_1(p^7,p^8)\Psi(p^3,p^8)}{|p|^{6s_1}}\mathcal{D}_2(s_1,\Psi',p^4) + \frac{|p|^6\phi(p^6)\Psi(p^5,p^8)}{|p|^{10s_1}}\mathcal{D}_2(s_1,\Psi'',1).$$

This may be directly verified up to Hilbert symbols (i.e. ignoring Hilbert symbols in the power reciprocity law in (5)) by using twisted multiplicativity to reconstruct $H(c_1,p^8;p^2,p)$ from $F^{(p)}(s_1;p^8)$ and $F^{(\ell)}(s_1;1)$. But to give a full accounting with Hilbert symbols one needs to verify that the "left-over" Hilbert symbols from repeated applications of reciprocity are precisely those required for the definitions of Ψ' and Ψ'' (again referring to Lemma 5.3 of [5]).

We now return to our general strategy of demonstrating the functional equation σ_1 as in (36). The function $Z_\Psi(s_1,s_2;p^2,p)$ as in (39) with fixed $c_2 = p^8$ yields $F(s_1;p^8)$ as above. We must verify that this portion of $Z_\Psi(s_1,s_2;p^2,p)$ is consistent with the desired global functional equation

$$Z_\Psi(s_1,s_2;p^2,p) \to |p^2|^{1-2s_1} Z_{\sigma_1\Psi}(1-s_1,s_1+s_2-1/2;p^2,p)$$

presented at the outset of this section. By Theorem 1,

$$\mathcal{D}_2(s_1, \Psi', p^4) \to |p^4|^{1-2s_1}\mathcal{D}_2(1 - s_1, \Psi', p^2)$$

and $|p|^{-6s_1-16s_2} \to |p|^{2-10s_1-16s_2}$ under σ_1. Similarly, $\mathcal{D}_2(s_1, \Psi'', 1) \to \mathcal{D}_2(1-s_1, \Psi'', p^2)$ and $|p|^{-10s_1-16s_2} \to |p|^{-2-6s_1-16s_2}$ under σ_1. Taken together, these calculations imply that

$$\frac{F(s_1; p^8)}{|p^8|^{2s_2}} \to |p^2|^{1-2s_1} \frac{F(1 - s_1; p^8)}{|p^8|^{2(s_1+s_2-1/2)}},$$

which is consistent with the global functional equation for Z_Ψ above.

Throughout the above analysis, we chose to restrict to the case where $c_2 = p^8$ to limit the complexity of the calculation. However, identical methods could be used to determine the global object for arbitrary choice of c_2 depending on the order of p dividing c_2, and hence verify the global functional equation for σ_1 in full generality.

Remark 2 With respect to the s_1 functional equation, it turns out to be quite simple to figure out which BZL-patterns contribute to a particular Kubota Dirichlet series appearing in $F(s_1; p^{k_2})$. All such BZL-patterns have identical top rows but differ in the bottom row entry. This entry increases as we increase k_1, as can be verified in our earlier table with $k_2 = 8$. However, as we will see in the next section, functional equations in s_2 and the respective Kubota Dirichlet series used in asserting them obey no such simple pattern.

6.2.2 The functional equation σ_2 with $k_1 = 3$

We now repeat the methods of the previous section to demonstrate a functional equation under σ_2. As we will show, it is significantly more difficult to organize the local contributions into linear combinations of Kubota Dirichlet series in terms of s_2. Once this is accomplished, the analysis proceeds along the lines of the previous section, so we omit further details.

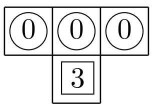
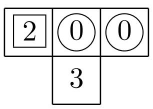
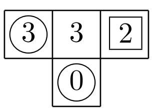
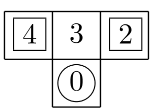
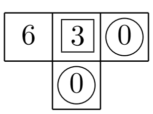
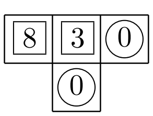
Let $c_1 = p^3$ be fixed. Mimicking our notation from the previous section, we now set

$$F(s_2; p^3) = \sum_{k_2} \frac{H^{(3)}(p^3, c_2; p^2, p)\Psi(p^3, c_2)}{|c_2|^{2s_2}}. \tag{43}$$

As in the previous section, the bulk of the difficulty lies in analyzing

$$F^{(p)}(s_2; p^3) = \sum_{k_2} \frac{H^{(3)}(p^3, p^{k_2}; p^2, p)\Psi(p^3, p^{k_2})}{|p|^{2k_2s_2}}.$$

Again referring to Figure 3, coefficients $H^{(3)}(p^{k_1}, p^{k_2}; p^2, p)$ with $k_1 = 3$ involve 9 different vertices and a total of 30 BZL-patterns, only six of which give nonzero contributions in the case when $n = 3$. In the table below, we list only those BZL-patterns yielding nonzero Gauss sums. The final column has again been computed from the third column, using the elementary properties of n^{th}-order Gauss sums mentioned in the previous subsection.

Δ	$(k_1, k_2) = k(\Delta)$	$G(\Delta)$	$G(\Delta)$ for $n = 3$
	$(3, 0)$	$g_2(p^2, p^3)$	$-\lvert p\rvert^2$
	$(3, 2)$	$g_1(p, p^2)\, g_2(p^4, p^3)$	$\lvert p\rvert g_2(1, p)\phi(p^3)$
	$(3, 5)$	$p^3\, g_1(p, p^2)\, g_1(p^4, p^3)$	$\lvert p\rvert^4 g_2(1, p)\phi(p^3)$
	$(3, 6)$	$g_1(p, p^2)\, g_1(p^3, p^4)\, g_2(p^4, p^3)$	$\lvert p\rvert^5 \phi(p^3)$
	$(3, 6)$	$g_1(p^7, p^6)\, g_2(p^2, p^3)$	$-\lvert p\rvert^2 \phi(p^6)$
	$(3, 8)$	$g_1(p^7, p^8)\, g_2(p^2, p^3)$	$-g_2(1, p)\lvert p\rvert^9$

According to the above table, we have

$$F^{(p)}(s_2;p^3) = -|p|^2\Psi(p^3,1) + \frac{|p|g_2(1,p)\phi(p^3)\Psi(p^3,p^2)}{|p|^{4s_2}} + \frac{|p|^4 g_2(1,p)\phi(p^3)\Psi(p^3,p^5)}{|p|^{10s_2}} + \frac{|p|^5\phi(p^3)\Psi(p^3,p^6)}{|p|^{12s_2}} - \frac{|p|^2\phi(p^6)\Psi(p^3,p^6)}{|p|^{12s_2}} - \frac{g_2(1,p)|p|^9\Psi(p^3,p^8)}{|p|^{16s_2}}. \tag{44}$$

By adding and subtracting certain necessary terms at vertices $(3,3)$ and $(3,5)$, and using the fact that $g_1(1,p)g_2(1,p) = |p|$ when $n = 3$, we find that $F^{(p)}(s_2;p^3)$ equals

$$\begin{aligned} &-|p|^2\Psi(p^3,1)\left[1 + \frac{\phi(p^3)}{|p|^{6s_2}}\frac{\Psi(p^3,p^3)}{\Psi(p^3,1)} + \frac{\phi(p^6)}{|p|^{12s_2}}\frac{\Psi(p^3,p^6)}{\Psi(p^3,1)} + \frac{g_2(1,p)|p|^7}{|p|^{16s_2}}\frac{\Psi(p^3,p^8)}{\Psi(p^3,1)}\right] \\ &+ \frac{g_2(1,p)|p|\,\phi(p^3)\Psi(p^3,p^2)}{|p|^{4s_2}}\left[1 + \frac{\phi(p^3)}{|p|^{6s_2}}\frac{\Psi(p^3,p^5)}{\Psi(p^3,p^2)} + \frac{g_1(1,p)p^3}{|p|^{8s_2}}\frac{\Psi(p^3,p^6)}{\Psi(p^3,p^2)}\right] \\ &+ \frac{|p|^2\phi(p^3)\Psi(p^3,p^3)}{|p|^{6s_2}}\left[1 + \frac{|p|g_2(1,p)}{|p|^{4s_2}}\frac{\Psi(p^3,p^5)}{\Psi(p^3,p^3)}\right]. \end{aligned} \tag{45}$$

After analyzing the terms in the bracketed sums, ignoring complications from the function Ψ as before, we have

$$F^{(p)}(s_2;p^3) = -|p|^2\Psi(p^3,1)\mathcal{D}_1^{(p)}(s_2,\Psi',p^7) + \frac{g_2(1,p)|p|\phi(p^3)\Psi(p^3,p^2)}{|p|^{4s_2}}\mathcal{D}_1^{(p)}(s_2,\Psi'',p^3) + \frac{|p|^2\phi(p^3)\Psi(p^3,p^3)}{|p|^{6s_2}}\mathcal{D}_1^{(p)}(s_2,\Psi''',p). \tag{46}$$

Arguing similarly to the previous section, one can use these local contributions to reconstruct the global Dirichlet series via twisted multiplicativity. The resulting objects satisfy the global functional equation for σ_2 as in (36).

6.2.3 The functional equation σ_2 with $k_1 = 6$

As a final example, the set of all $H^{(3)}(p^{k_1},p^{k_2};p^2,p)$ with $k_1 = 6$ involves 7 support vertices and 18 BZL-patterns. In the case $n = 3$, however, only four of the BZL-patterns have non-zero Gauss sum products associated to them. These are listed in the table below.

Δ	$k(\Delta)$	$G(\Delta)$	$G(\Delta)$ for $n=3$
$\begin{array}{ccc} 6 & \boxed{3} & \textcircled{0} \\ & \boxed{3} & \end{array}$	$(6,6)$	$g_1(p^7,p^6)\,g_2(p^2,p^3)\,g_2(p^2,p^3)$	$\lvert p\rvert^4\,\phi(p^6)$
$\begin{array}{ccc} \boxed{4} & 3 & \boxed{2} \\ & \boxed{3} & \end{array}$	$(6,6)$	$g_1(p^1,p^2)\,g_1(p^3,p^4)\,g_2(p^2,p^3)\,g_2(p^4,p^3)$	$-\lvert p\rvert^7\,\phi(p^3)$
$\begin{array}{ccc} \boxed{8} & \boxed{3} & \textcircled{0} \\ & 3 & \end{array}$	$(6,8)$	$g_1(p^7,p^8)\,g_2(p^2,p^3)\,g_2(p^4,p^3)$	$-\lvert p\rvert^9 g_2(1,p)\phi(p^3)$
$\begin{array}{ccc} 6 & \boxed{5} & \boxed{2} \\ & \boxed{1} & \end{array}$	$(6,8)$	$g_1(p^1,p^2)\,g_1(p^7,p^6)\,g_2(1,p)\,g_2(p^4,p^5)$	$\lvert p\rvert^{11} g_2(1,p)\phi(p^6)$

Upon first inspection, it is unclear how to package the Gauss sum products neatly into p-parts of Kubota Dirichlet series, as in the previous examples. However, the two nonzero terms at $(6,6)$ cancel each other out when $n=3$, as do the two nonzero terms at $(6,8)$. This seems like a very complicated way to write 0, but we remind the reader that the definition in terms of Gauss sums is "uniform" in n, in the sense that only the order of the multiplicative character in the Gauss sum changes. For other n, the p-part $H^{(n)}(p^{k_1},p^{k_2};p^2,p)$ with $k_1=6$ will have the same 18 products of Gauss sums, four of which are as shown in the third column of the table above. However, the evaluations as in the last column of the table depend on the choice of n and result in a different organization as Kubota Dirichlet series.

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