**April 12, 12:00 pm**

# 1 Introduction

The study of modular forms is typically reserved for graduate students, because the amount of background needed to fully appreciate many of the constructions and methods is rather large. However, it is possible to get a first look at modular forms without relying too heavily on the theory of complex analysis, harmonic analysis, or differential geometry. In some sense, we've been doing this all semester – using identities which can be understood more generally in the language of modular forms to prove classic problems in number theory that were only completely resolved after years of work beyond Ramanujan by a host of talented mathematicians.

In the remainder of this course, we'll be exploring the theory of modular forms equipped with a large number of examples coming from the generating functions we've been analyzing all semester.

# 2 The Modular Group

In our recent study of elliptic functions, we were led to consider the set of matrices

$$GL(2,\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a,b,c,d \in \mathbb{Z}, ad-bc = \pm 1 \right\}$$

which related any two pairs of complex numbers $(\omega_1, \omega_2)$ generating the period module $M$. Furthermore, we learned that to any lattice, we may choose a basis $\langle \omega_1, \omega_2 \rangle$ with $\tau = \frac{\omega_2}{\omega_1} \in \mathbb{C}$ uniquely determined by the criteria

$$\tau : \mathrm{Im}(\tau) > 0, |\tau| \geq 1, \frac{-1}{2} < \mathrm{Re}(\tau) \leq \frac{1}{2}$$

(which, in turn, almost uniquely determined the pair $\omega_1, \omega_2$).

In reviewing the definition for modular forms, we typically state definitions with respect to the group

$$SL(2,\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a,b,c,d \in \mathbb{R}, ad-bc = 1 \right\}.$$

There is an action of $SL(2,\mathbb{R})$ on $\mathbb{C}$ given by

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{R}) : \quad \gamma(z) = \frac{az+b}{cz+d}$$

It is sometimes convenient to extend this action to the set $\mathbb{C} \cup \{\infty\}$ by letting $\gamma(\infty) = \frac{a}{c}$ and $\gamma(\frac{d}{c}) = \infty$. Moreover, one checks (EXERCISE, if you haven't tried it before) that with notation for $\gamma$ as above,

$$\mathrm{Im}(\gamma(z)) = \frac{\mathrm{Im}(z)}{|cz+d|^2}.$$

Hence, we see that the action takes $\mathbb{R} \cup \infty$ to itself, and stabilizes the upper and lower half planes. Let $\mathcal{H}$ denote the upper half plane $\{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$. Moreover the matrix $-I = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ acts trivially on $\mathcal{H}$, i.e. fixes all elements.

**Proposition 1** *The group $PSL(2,\mathbb{R}) := SL(2,\mathbb{R})/\{\pm I\}$ acts faithfully on $\mathcal{H}$.*

Recall that a *faithful* group action is one for which no non-indentity element of the group fixes all elements of the set. We leave the proof of this proposition as an EXERCISE.

The "modular group" $G$ is the subgroup $SL(2,\mathbb{Z})/\{\pm I\}$ in $PSL(2,\mathbb{R})$, consisting of matrices with coefficients in $\mathbb{Z}$ up to equivalence by $\pm I$. Technically, we should denote elements of this quotient group as cosets, but typically no confusion will arise by continuing to use the matrix representation of $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in $SL(2,\mathbb{R})$ to denote elements of $G$. (It is important that $SL(2,\mathbb{Z})$ is a discrete subgroup of $SL(2,\mathbb{R})$, that is a topological group with the discrete topology. The theory of modular forms can be presented for arbitrary discrete groups of $SL(2,\mathbb{R})$ with some additional complications. For more facts about discrete subgroups of $SL(2,\mathbb{R})$, see Shimura's book.

# 3 The Fundamental Domain for $G$

In this section, we show that the domain $D = \{z \in \mathcal{H} \mid |z| \geq 1, |\mathrm{Re}(z)| \leq 1/2\}$ is a fundamental domain for the action of $G$ on $\mathcal{H}$ (and explain what is meant by this term "fundamental domain"). To this end, consider the matrices

$$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} : S(z) = \frac{-1}{z} \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} : T(z) = z+1$$

which satisfy the relations

$$S^2 = I, \quad (ST)^3 = I.$$

We will prove the following two main results in this section:

**Theorem 1** *$D$ is a fundamental domain for the action of $G$ on $\mathcal{H}$. That is, by way of definition,*

1. *For every $z \in \mathcal{H}$ there exists $\gamma \in G$ such that $\gamma(z) \in D$.*
2. *Given two points $z, z' \in D$ with $z = \gamma(z')$, then either $Re(z) = \pm\frac{1}{2}$ and $z = z' \pm 1$ OR $|z| = 1$ and $z = \frac{-1}{z'}$.*
3. *To each $z \in D$, let $Stab(z) = \{\gamma \in G \mid \gamma(z) = z\}$. Then $Stab(z) = \{I\}$ for all $z \in D$ UNLESS*
   - *$z = i$, then $Stab(z) = \langle S \rangle$ of order 2.*
   - *$z = e^{2\pi i/3}$, then $Stab(z) = \langle ST \rangle$ of order 3.*
   - *$z = e^{\pi i/3}$, then $Stab(z) = \langle TS \rangle$ of order 3.*

**Theorem 2** *The group $G$ is generated by $S$ and $T$, i.e. every element $\gamma \in G$ can be written as a word in $S$ and $T$ of form*

$$\gamma = T^{n_1} S T^{n_2} S \cdots S T^{n_k}$$

*for some choice of integers $n_i$ (though the representation is clearly not unique according to the relations mentioned above).*

We will use Serre's proof of these facts, which proves both theorems at the same time. Before launching into the proof, a simple example will serve to illustrate an alternate proof of Theorem 2. Given any matrix, say

$$\gamma = \begin{pmatrix} 4 & 9 \\ 11 & 25 \end{pmatrix}$$

we seek to represent $\gamma$ in terms of $S$ and $T$. Note

$$\gamma T^n = \begin{pmatrix} 4 & 9 \\ 11 & 25 \end{pmatrix} \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 4 & 4n+9 \\ 11 & 11n+25 \end{pmatrix}, \gamma S = \begin{pmatrix} 4 & 9 \\ 11 & 25 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 9 & -4 \\ 25 & -11 \end{pmatrix}.$$

Hence, we may choose $n$ to reduce the size of coefficients in the matrix $\gamma T^n$. In particular, noting the way in which $S$ switches the columns of the matrix, we may choose $n$ (say $n = -2$ in our case) so that $|d| < |c|$. Then

$$\gamma T^{-2} = \begin{pmatrix} 4 & 1 \\ 11 & 3 \end{pmatrix}. \text{ Now } \gamma T^{-2} S = \begin{pmatrix} 4 & 1 \\ 11 & 3 \end{pmatrix} = \begin{pmatrix} 4 & 1 \\ 11 & 3 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & -4 \\ 3 & -11 \end{pmatrix}.$$

And we may continue this process, now reducing 11 mod 3 by multiplying by say $T^4$ (though $T^3$ would work as well). Carrying this example to the bitter end, we may obtain $\gamma = ST^{-3}ST^{-4}ST^2$.

EXERCISE: Turn the ideas used in this example into a rigorous general proof of Theorem 2.

**Proof** (of Theorems 1 and 2): Let $G'$ be the subgroup of $G$ generated by $S$ and $T$. We will first show part 1 of Theorem 1 by demonstrating an element of $\gamma' \in G'$ such that $\gamma'(z) \in D$.

There exists a $\gamma \in G'$ such that $\mathrm{Im}(\gamma(z))$ is maximal, because the number of pairs of integers $(c, d)$ with $|cz + d| < k$ for any given number $k$ is finite, and

$$\mathrm{Im}(\gamma(z)) = \frac{\mathrm{Im}(z)}{|cz + d|^2}. \tag{1}$$

Now choose an $n$ so that $T^n(\gamma(z))$ is shifted into the vertical strip between $-1/2$ and $1/2$. But then $z' = T^n(\gamma(z)) \in D$, since if $|z'| < 1$, then $S(z') = -1/z'$ would have a larger imaginary part than $z'$, contradicting the maximality of the imaginary part of $\gamma(z)$.

For parts 2 and 3 of Theorem 1, suppose that given $\gamma \in G$ and $z \in D$, $\gamma(z) \in D$ as well. As the pairs $(z, \gamma)$ and $(\gamma(z), \gamma^{-1})$ play symmetric roles here, we may assume without loss of generality that $\mathrm{Im}(\gamma(z)) \geq \mathrm{Im}(z)$, which implies from (1) that $|cz + d| \leq 1$. Remembering that $z \in D$, then $|cz + d| \leq 1$ for very few choices of $c$ and $d$, in particular only if $c = 0, 1, -1$ and we separate into three cases accordingly. If $c = 0$, then $d = \pm 1$ so $\gamma = \begin{pmatrix} \pm 1 & b \\ 0 & \pm 1 \end{pmatrix}$, i.e. $\gamma(z) = z \pm b$. But if $\gamma(z)$ is also assumed in $D$ which has width 1, then either $b = 0$ so $\gamma = \pm I$ or $b = \pm 1$, and then $\mathrm{Re}(z), \mathrm{Re}(\gamma(z))$ are $-1/2$ and $1/2$ (or vice versa). The cases $c = \pm 1$ follow similarly.

EXERCISE: Finish the remaining cases $c = \pm 1$ in the above proof to complete the proof of theorem 1. (Note: so far, the $c = 0$ case provided no non-identity matrices that stabilized a point in $D$. As written in the theorem, there are points with non-trivial stabilizers, so they must occur for $c = \pm 1$.)

Finally, it remains to prove that $G'$, our group generated by $S$ and $T$, is $G$. Pick any point $z_0$ in the interior of $D$. For any $\gamma \in G$, we must show $\gamma \in G'$. If $z = \gamma(z_0)$, then there exists a $\gamma' \in G'$ such that $\gamma'(z) = \gamma'\gamma(z_0) \in D$ (by our above argument for Theorem 1, part 1). But since $z_0$ was chosen in the interior of $D$, and $z_0, \gamma'\gamma(z_0)$ are both in $D$, then part 2 of Theorem 1 implies that $\gamma'\gamma = I$ and hence $\gamma \in G'$. $\square$

Serre also notes that one can prove the slightly stronger statement that the only relations on elements in $G$ are those generated by $S^2 = 1$ and $(ST)^3 = 1$.

# 4 Modular Functions, Modular Forms

**Definition 1** *For a given integer $k$, we say a function $f$ is weakly modular of weight $2k$ if $f(z)$ is meromorphic on $\mathcal{H}$ and satisfies*

$$f(z) = (cz+d)^{-2k} f\left(\frac{az+b}{cz+d}\right) \quad \textit{for all} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z})$$

Note that weakly modular functions of odd weight are all 0, since taking $\gamma = -I$ in the above definition would give $f(z) = (-1)^{2k+1} f(z)$. Equivalently, we could have required the transformation property to be well-defined for $G = SL(2,\mathbb{Z})/\{\pm I\}$.

The definition is natural because $d(\gamma(z))/dz = (cz+d)^{-2}$, so we may rewrite the transformation property as

$$\frac{f(\gamma(z))}{f(z)} = \left(\frac{d(\gamma(z))}{dz}\right)^{-k}, \quad \text{i.e. } f(\gamma(z))d(\gamma(z))^k = f(z)dz^k.$$

In the language of differential forms, this means the $k$-form $f(z)dz^k$ is invariant under the action of $G$.

Finally, we remark that since $G$ is generated by $S$ and $T$, it suffices to check that $f$ is invariant under these two transformations. That is, a meromorphic function $f$ is weakly modular of weight $2k$ if and only if

$$f(z+1) = f(z) \quad \text{and} \quad f(-1/z) = z^{2k} f(z).$$

In particular, if $f(z+1) = f(z)$, then our function is simply periodic and has a Fourier expansion in terms of powers of $q = e^{2\pi i z}$ which is meromorphic in the punctured disk $0 < |q| < 1$. If this can be extended to a meromorphic function at the origin, then we say that $f$ is "meromorphic at infinity." This term comes from the theory of Riemann surfaces where we think of $e^{2\pi i z}$ as a change of coordinates taking the point $i\infty$ to the origin. For us, we simply note that this means, practically speaking, that the Fourier expansion can be given as a Laurent series in $q$. If this Laurent expansion has no negative powers, we say $f$ is "holomorphic at infinity."

**Definition 2** *A weakly modular function $f$ which is holomorphic everywhere (including $\infty$) is called a "modular form." If the function is 0 at infinity (that is, the Laurent expansion in $q$ has $a_0 = 0$) then we say $f$ is a "cusp form."*

## 5 Examples: Eisenstein Series

We've seen Eisenstein series defined in several ways so far this semester. Most recently, they were associated to lattices, but typically in the theory of modular forms they are associated to a complex variable in the upper half plane. We first clarify the translation between the two types of functions.

In our unit on elliptic functions, we identified the set of lattices in $\mathbb{C}$ as a real vector space spanned by pairs of complex numbers $\omega_1, \omega_2$ with say $\mathrm{Im}(\omega_1/\omega_2) > 0$. Then the set of all lattices $\mathcal{R}$ may be identified with the set of all such pairs

$$L = \{(\omega_1, \omega_2) \mid \mathrm{Im}(\omega_1/\omega_2) > 0\}$$

where we quotient out by the action of $SL(2, \mathbb{Z})$, which accounts for all possible changes of basis. Moreover, $\lambda \in \mathbb{C}^\times = \mathbb{C} - \{0\}$ acts on $L$ by $\lambda : (\omega_1, \omega_2) \mapsto (\lambda\omega_1, \lambda\omega_2)$, so we may identify $L/\mathbb{C}^\times$ with $\mathcal{H}$ by the map $(\omega_1, \omega_2) \mapsto z = \omega_1/\omega_2$. Note that the action of $SL(2, \mathbb{Z})$ on $L$ translates to the usual action of $G = SL(2, \mathbb{Z})/\{\pm I\}$ on $\mathcal{H}$. So we have

**Proposition 2** *The map $(\omega_1, \omega_2) \mapsto \omega_1/\omega_2$ gives (after passing to the quotient by $SL(2, \mathbb{Z})$) a bijection between $\mathcal{R}/\mathbb{C}^\times$ and $\mathcal{H}/G$.*

Let $F$ be a complex-valued function on $\mathcal{R}$, the space of lattices. We say $F$ is of weight $2k$ if

$$F(\lambda\Gamma) = \lambda^{-2k} F(\Gamma) \quad \text{for all lattices } \Gamma \in \mathcal{R}, \text{ all } \lambda \in \mathbb{C}^\times.$$

In particular if $\Gamma = \Gamma(\omega_1, \omega_2)$, the lattice generated by $\omega_1, \omega_2$, we may write $F$ as a function of the basis elements and

$$F(\lambda\omega_1, \lambda\omega_2) = \lambda^{-2k} F(\omega_1, \omega_2)$$

and setting $\lambda = \omega_2^{-1}$ shows that $\omega_2^{2k} F(\omega_1, \omega_2)$ depends only on $z = \omega_1/\omega_2$, so we may rewrite

$$F(\omega_1, \omega_2) = \omega_2^{-2k} f(\omega_1/\omega_2) \quad \text{for some } f : \mathcal{H} \to \mathbb{C}$$

and then $f(z)$ will be a modular function of weight $2k$ in terms of $z \in \mathcal{H}$.

In this vein, we may begin with an Eisenstein series associated to any lattice:

$$G_k(\Gamma) = \sum_{v \neq (0,0) \in \Gamma} \frac{1}{v^{2k}}, \quad k > 1$$

where we have taken $k > 1$ to guarantee the absolute convergence of the series. In terms of a basis $\omega_1, \omega_2$ for the lattice we may rewrite this as

$$G_k(\omega_1, \omega_2) = \sum_{(m,n)\neq(0,0)} \frac{1}{(m\omega_1 + n\omega_2)^{2k}}$$

which is then related by a power of $\omega_2^{2k}$ to

$$G_k(z) = \sum_{(m,n)\neq(0,0)} \frac{1}{(mz + n)^{2k}}$$

**Proposition 3** *For $k > 1$, the Eisenstein series $G_k(z)$ is a modular form of weight $2k$. Moreover, this function can be extended to $\mathcal{H} \cup \{\infty\}$ with $G_k(\infty) = 2\zeta(2k)$, where $\zeta$ is the usual Riemann zeta function.*

**Proof** It is clear that the series is a weakly modular function of weight $2k$, as the series converges absolutely for $k > 1$ and hence the modularity property reduces to a simple change of variables in the sum, together with a rearrangement of summands. So it is left to show that the Eisenstein series defines a holomorphic function on $H \cup \{\infty\}$.

Suppose $z \in D$. Then

$$\begin{aligned} |mz + n|^2 &= m^2 z\bar{z} + 2mn\mathrm{Re}(z) + n^2 \\ &\geq m^2 - mn + n^2 = |m\rho - n|^2 \quad \text{where } \rho = e^{2\pi i/3} \end{aligned}$$

But $\sum_{(m,n)\neq(0,0)} 1/|m\rho - n|^{2k}$ converges, so $G_k(z)$ converges uniformly in $D$. Performing the same calculation for $G_k(\gamma^{-1}z)$ with $\gamma \in G$ shows the same is true on each of the sets $\gamma(D)$ which cover the entire upper half-plane $\mathcal{H}$. It remains to show that $G_k$ is holomorphic at infinity, i.e. that $G_k$ has a limit as $\mathrm{Im}(z) \to \infty$. We may take a limit running over $z \in D$ (according to the modularity property of $G_k$) and by uniform convergence in $D$, we can take the limit term by term in the sum. For $m \neq 0$, these terms give 0, for $m = 0$, we get $1/n^{2k}$. Hence the limit exists for $G_k(z)$ and is equal to $2\zeta(2k)$ upon summing over all integers $n$. $\square$

We now turn to the Fourier expansions of the Eisenstein series. Our proof is motivated by the construction of a Laurent expansion in $z$ for the Weierstrass $\wp$ function. That expansion was in $z$, while this will be a Fourier expansion in $q = e^{2\pi i z}$, but the principle is the same: use term by term differentiation of a uniformly convergent series after expressing an initial function as a (Laurent or Fourier) expansion.

We know we want this term by term differentiation to result in something of the form $\frac{1}{nz+m}^{2k}$. The $nz$ can be simplified by change of variable $z \mapsto z/n$, so we want a function expressed as a sum of $\frac{1}{z+m}$, and whose Fourier expansion is known to us. Searching among trig functions, we find

$$\pi \cot(\pi z) = \frac{1}{z} + \sum_{m=1}^{\infty} \left( \frac{1}{z+m} + \frac{1}{z-m} \right)$$

One way to see this is via logarithmic differentiation of the product formula for $\sin z$:

$$\sin z = z \prod_{n=1}^{\infty} \left( 1 - \frac{z^2}{n^2 \pi^2} \right).$$

On the other hand, we may write

$$\pi \cot(\pi z) = \pi \frac{\cos(\pi z)}{\sin(\pi z)} = i\pi \frac{q+1}{q-1} = i\pi - \frac{2\pi i}{1-q} = i\pi - 2\pi i \sum_{n=0}^{\infty} q^n.$$

Combining the two expressions for $\pi \cot(\pi z)$ and differentiating both sides $2k$ times with respect to $z$, we obtain the following expression (valid for $k \geq 1$):

$$\sum_{m \in \mathbb{Z}} \frac{1}{(m+z)^{2k}} = \frac{1}{(2k-1)!} (-2\pi i)^{2k} \sum_{n=1}^{\infty} n^{2k-1} q^n \tag{2}$$

Now we are prepared to assert:

**Proposition 4** *For every integer $k \geq 2$,*

$$G_k(z) = 2\zeta(2k) + 2 \frac{(2\pi i)^{2k}}{(2k-1)!} \sum_{n=1}^{\infty} \sigma_{2k-1}(n) q^n$$

**Proof** Recalling the definition,

$$G_k(z) = \sum_{(m,n) \neq (0,0)} \frac{1}{(nz+m)^{2k}} = 2\zeta(2k) + 2 \sum_{n=1}^{\infty} \sum_{m \in \mathbb{Z}} \frac{1}{(nz+m)^{2k}}$$

where we've taken the sum over integers $n$ and split up the terms $n = 0$ and $n \neq 0$ into the pieces above. Now applying (2) with $nz$ instead of $z$, we get

$$G_k(z) = 2\zeta(2k) + \frac{2(2\pi i)^{2k}}{(2k-1)!} \sum_{d=1}^{\infty} \sum_{a=1}^{\infty} d^{2k-1} q^{ad} = 2\zeta(2k) + \frac{2(2\pi i)^{2k}}{(2k-1)!} \sum_{n=1}^{\infty} \sigma_{2k-1}(n) q^n,$$

as desired. $\square$

Sometimes it is convenient to normalize the constant term to be 1. Letting $E_k(z) = G_k(z)/2\zeta(2k)$, gives

$$E_k(z) = 1 + \gamma_k \sum_{n=1}^{\infty} \sigma_{2k-1}(n) q^n$$

with

$$\gamma_k = (-1)^k \frac{4k}{B_k}, \quad B_k \text{ the } k\text{th Bernoulli number:} \frac{x}{e^x - 1} = 1 - \frac{x}{2} + \sum_{k=1}^{\infty} (-1)^{k+1} B_k \frac{x^{2k}}{(2k)!}$$

To achieve this expression, we used Euler's result on the special values of the zeta function at even integers:

$$\zeta(2k) = \frac{2^{2k-1}}{(2k)!} B_k \pi^{2k},$$

noting that all factors other than $B_k$ in this formula nicely cancel with factors in front of the Fourier coefficients of $G_k$, leaving $\gamma_k$ as a rather simple expression.

# 6 Product Identities and the Dedekind Eta Function

Defined by Dedekind in 1877, as usual we take $\tau \in \mathcal{H}$ and define the function

$$\eta(\tau) = e^{2\pi i \tau/24} \prod_{n=1}^{\infty} (1 - e^{2\pi i n \tau})$$

Substituting in $q = e^{2\pi i \tau}$, then $\tau \in \mathcal{H}$ implies $|q| < 1$ and

$$\eta(q) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n) = q^{1/24} (q; q)_{\infty}$$

which is non-zero and converges absolutely for $|q| < 1$.

As we will subsequently prove, the $\eta$ function is related to the discriminant function by $\Delta(\tau) = (2\pi)^{12} \eta^{24}(\tau)$. (We've mentioned both the infinite product definition of $\Delta(\tau)$ and it's description as cusp form (i.e. $a_0 = 0$ in the Fourier expansion) defined as a combination of Eisenstein series of weights 4 and 6. But we've never PROVED that the two definitions are consistent.)

Let's examine the effect of the basic transformations $S$ and $T$ that generate $SL(2,\mathbb{Z})$. For $T:\tau\mapsto\tau+1$, we have

$$\eta(\tau+1)=e^{2\pi i(\tau+1)/24}\prod_{n=1}^{\infty}(1-e^{2\pi in(\tau+1)})=e^{\pi i/12}\eta(\tau)$$

Modular forms are supposed to be completely invariant under translation, according to the definition. In particular, $\eta^{24}(\tau)$ will have period 1. It is slightly trickier to determine the effect of $S:\tau\mapsto\frac{-1}{\tau}$.

**Theorem 3** *For $\tau\in\mathcal{H}$,*

$$\eta\left(\frac{-1}{\tau}\right)=(-i\tau)^{1/2}\eta(\tau)$$

In Apostol's book, he offers several proofs of this fact. We start with a proof due to Siegel using a bit of complex analysis. We prove the result for $\tau=iy$, i.e. $\tau$ along the positive imaginary axis in $\mathcal{H}$ and then extend the result to the entire complex plane via a theorem on analytic continuation.

**Proof** For $\tau=iy$, the transformation formula becomes

$$\eta(i/y)=y^{1/2}\eta(iy).$$

Now taking logs to convert the product expansion to a sum, we must show

$$\log\eta(i/y)-\log\eta(iy)=\frac{1}{2}\log y. \tag{3}$$

But then

$$\begin{aligned}\log\eta(iy) &= -\frac{\pi y}{12}+\sum_{n=1}^{\infty}\log(1-e^{-2\pi ny})=-\frac{\pi y}{12}-\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{e^{-2\pi mny}}{m}\\ &= -\frac{\pi y}{12}-\sum_{m=1}^{\infty}\frac{1}{m}\frac{e^{-2\pi my}}{1-e^{2\pi my}}=-\frac{\pi y}{12}+\sum_{m=1}^{\infty}\frac{1}{m}\frac{1}{1-e^{2\pi my}}\end{aligned}$$

Substituting this back into (3), we must show that

$$\sum_{m=1}^{\infty}\frac{1}{m}\frac{1}{1-e^{2\pi my}}-\sum_{m=1}^{\infty}\frac{1}{m}\frac{1}{1-e^{2\pi m/y}}-\frac{\pi}{12}\left(y-\frac{1}{y}\right)=-\frac{1}{2}\log y. \tag{4}$$

For fixed real $y>0$ and integer $n\geq 1$, define

$$F_n(z)=-\frac{1}{8z}\cot(\pi i(n+1/2)z)\cot\left(\frac{\pi(n+1/2)z}{y}\right).$$

We will study the contour integral of $F_n(z)$ around the parallelogram formed with vertices $y, i, -y, -i$, traversed from starting point $y$ (drawing a quick picture might help here). By Cauchy's Integral Theorem, we must determine the location of the poles in $z$ lying inside this contour, and then the residue of the function $F_n(z)$ at each of these points.

The non-zero poles coming from the first cot function are at $z = ik/(n+1/2)$ for $k = \pm 1, \cdots, \pm n$, those from the second cot in the product are at $z = ky/(n+1/2)$ for $k = \pm 1, \cdots, \pm n$. Finally, there's a triple pole at the origin $z = 0$ with residue $i(y - \frac{1}{y})/24$. The residue at $z = ik/(n+1/2)$ for each $k$ is

$$\frac{1}{8\pi k} \cot \pi i k y$$

which is an even function of $k$. So summing over the $2n$ non-zero residues gives:

$$\sum_{k=-n,\, k\neq 0}^{n} \mathrm{Res}_{z=ik/(n+1/2)} F_n(z) = 2 \sum_{k=1}^{n} \frac{1}{8\pi k} \cot \pi i k y.$$

Recalling that

$$\cot i\theta = \frac{\cos i\theta}{\sin i\theta} = i\frac{e^{-\theta} + e^{\theta}}{e^{-\theta} - e^{\theta}} = -i\frac{e^{2\theta} + 1}{e^{2\theta} - 1} = \frac{1}{i}\left(1 - \frac{2}{1 - e^{2\theta}}\right),$$

we may rewrite the above sum using $\theta = \pi k/y$ to get

$$\sum_{k=-n,\, k\neq 0}^{n} \mathrm{Res}_{z=ik/(n+1/2)} F_n(z) = \frac{1}{4\pi i} \sum_{k=1}^{n} \frac{1}{k} - \frac{1}{2\pi i} \sum_{k=1}^{n} \frac{1}{k} \frac{1}{1 - e^{2\pi k/y}}.$$

By identical methods for the other set of non-zero residues, we have

$$\sum_{k=-n,\, k\neq 0}^{n} \mathrm{Res}_{z=ky/(n+1/2)} F_n(z) = \frac{i}{4\pi} \sum_{k=1}^{n} \frac{1}{k} - \frac{i}{2\pi} \sum_{k=1}^{n} \frac{1}{k} \frac{1}{1 - e^{2\pi k y}}.$$

In other words, taking $2\pi i$ times the sum of ALL residues inside $C$ gives an expression which, upon taking the limit as $n \to \infty$ (which does not alter the contour $C$), agrees with the left hand side of (4). By Cauchy's integral theorem, this should equal the contour integral, so it remains to show that:

$$\lim_{n\to\infty} \int_C F_n(z)dz = -\frac{1}{2}\log y$$

Because $F_n$ is uniformly bounded on $C$ for all $n$ (we chose a fixed $y > 0$), we can apply a bounded convergence theorem to change the order of the limit and the integration. Moreover,

$$\lim_{n\to\infty} F_n(z) = -\frac{1}{8z}\lim_{n\to\infty} \cot(\pi i(n+1/2)z)\cot\left(\frac{\pi(n+1/2)z}{y}\right)$$

Remembering that $\cot z = i\frac{e^{2iz}+1}{e^{2iz}-1}$, then this limit will be $\pm 1$ according to whether the real and imaginary parts of $z$ have the same sign. In conclusion, $\lim F_n(z) = 1/8z$ for points $z$ on $C$ in the first and third quadrant, and $-1/8z$ for points in the second and fourth quadrant on $C$. Putting it all together, we have

$$\begin{aligned}
\lim_{n\to\infty}\int_C F_n(z)dz &= \int_C \lim_{n\to\infty} F_n(z)dz \\
&= \frac{1}{8}\left(\int_y^i \frac{dz}{z} - \int_i^{-y}\frac{dz}{z} + \int_{-y}^{-i}\frac{dz}{z} - \int -i^y \frac{dz}{z}\right) \\
&= \frac{1}{4}\left(\int_y^i \frac{dz}{z} - \int_{-i}^{y}\frac{dz}{z}\right) = -\frac{1}{2}\log y.
\end{aligned}$$

□

# 7 Product Expansion for $\Delta(\tau)$

Let us take, as our definition of $\Delta(\tau)$,

$$\Delta(\tau) = g_2^3(\tau) - 27g_3^2(\tau)$$

where $g_2$ and $g_3$ are the weight 4 and 6 Eisenstein series, appropriately normalized:

$$g_2(\tau) = 60\sum_{(m,n)\neq(0,0)}\frac{1}{(m\tau+n)^4}, \quad g_3(\tau) = 140\sum_{(m,n)\neq(0,0)}\frac{1}{(m\tau+n)^6}.$$

We chose this relation for $\Delta(\tau)$ because we wanted to exhibit the first example of a cusp form for $SL(2,\mathbb{Z})$, a modular form whose expansion at infinity has constant term 0. It is clear that we can do this, provided we can find two distinct modular forms of the same weight. Both $g_2^3$ and $g_3^2$ have weight 12, so $\Delta(\tau)$ is a modular form of weight 12 as well.

You might be wondering at this point whether cusp forms of smaller weight exist for $SL(2,\mathbb{Z})$ and whether modular forms of weight less than 4 exist for $SL(2,\mathbb{Z})$. Or whether we can form modular forms for other groups. We'll get to these questions in subsequent sections. In this section, we finally prove a product expansion for $\Delta$ as defined above.

**Theorem 4** *Let $\tau \in \mathcal{H}$, the complex upper half plane. Let $q = e^{2\pi i\tau}$. Then*

$$\Delta(\tau) = (2\pi)^{12}\eta^{24}(\tau) = (2\pi)^{12} q \prod_{n=1}^{\infty}(1-q^n)^{24}.$$

From our investigations in the last section, we see that $\eta^{24}(\tau)$ will be a cusp form of weight 12 on $SL(2,\mathbb{Z})$ according to the transformation properties under $S$ and $T$. So if we knew that the space of cusp forms of weight 12 was one dimensional (as a complex vector space), then we would only need to determine the constant relating $\Delta(\tau)$ and $\eta^{24}(\tau)$. The proof we will give has similar techniques in common to certain proofs of the dimension of the spaces of modular forms and cusp forms.

**Proof** Consider the function $f(\tau) = \Delta(\tau)/\eta^{24}(\tau)$. Because both the numerator and denominator are weight 12 modular forms, $f(\tau) = f(\gamma(\tau))$ for all $\gamma \in G$. Moreover, $f$ is analytic and non-zero in $\mathcal{H}$ because, as shown in our unit on elliptic functions, $\Delta$ is non-zero and analytic on $\mathcal{H}$ and the product expansion for $\eta$ (together with logarithmic differentiation) shows that $\eta$ never vanishes on $\mathcal{H}$. It remains to analyze the behavior of $f$ at $\infty$. Note

$$\eta^{24}(\tau) = q\prod_{n=1}^{\infty}(1-q^n)^{24} = q(1+I(q))$$

where $I(q)$ is some power series in $q$ with integer coefficients. In particular, $\eta^{24}$ has a first order zero at $q = 0$ (i.e. at infinity). Using the $q$-expansions for $g_2$ and $g_3$ presented in an earlier section, we find

$$\Delta(\tau) = (2\pi)^{12}\sum_{n=1}^{\infty}\tau(n)q^n = (2\pi)^{12}q(1+I'(q)),$$

where $I'(q)$ is another integral power series in $q$. Hence

$$f(\tau) = \frac{\Delta(\tau)}{\eta^{24}(\tau)} = \frac{(2\pi)^{12}q(1+I'(q))}{q(1+I(q))} = (2\pi)^{12}(1+I''(q))$$

for some $I''(q)$, and hence $f$ is analytic and non-zero at infinity. But then $f(\tau)$ is a modular function of weight 0 which never takes the value 0, hence must be constant. Our calculation at infinity shows this constant must be $(2\pi)^{12}$. $\square$

# 8 The Eisenstein series $G_1$

Previously, we had noted that $G_k(z)$ defined by

$$G_k(z) = \sum_{(m,n)\neq(0,0)} \frac{1}{(nz+m)^{2k}}$$

defines an absolutely convergent function on $\mathcal{H}$ with expansion at infinity

$$2\zeta(2k) + \frac{2(2\pi i)^{2k}}{(2k-1)!} \sum_{n=1}^{\infty} \sigma_{2k-1}(n) q^n.$$

The series defined as a sum over pairs of integers fails to converge absolutely for $k = 1$, so we didn't see a way to define an Eisenstein series of weight 2. However, the Fourier expansion for $G_k(z)$ does make sense for $k = 1$. Thus, we define

$$G_1(z) := 2\zeta(2) + 2(2\pi i)^2 \sum_{n=1}^{\infty} \sigma_1(n) q^n,$$

where as usual, $q = e^{2\pi i z}$. (Check that the definition above gives an absolutely convergent power series for $|q| < 1$, and hence defines an analytic function on the upper half plane $\mathcal{H}$.) Moreover, by taking as our definition a series expansion in $e^{2\pi i z}$, we immediately have that $G_1(z+1) = G_1(z)$. As for the other generator $S$ of $SL(2,\mathbb{Z})$, we'd like to show

$$G_1\left(\frac{-1}{z}\right) = z^2 G_1(z)$$

and hence $G_1$ is a weight 2 modular form. (Note, we can't just rearrange the series in acting by $S$ because the double sum is not absolutely convergent.) In fact, what turns out to be true is that

$$G_1\left(\frac{-1}{z}\right) = z^2 G_1(z) - 2\pi i z,$$

so $G_1$ is not a modular form of weight 2. However, these rather nice transformation properties often make $G_1$ useful for proofs involving modular forms. We can also use it to derive a second proof of the transformation property of $\eta$ under $S$.

**Theorem 5** *For any $z \in \mathcal{H}$,*

$$G_1\left(\frac{-1}{z}\right) = z^2 G_1(z) - 2\pi i z$$

The typical way of proving this is to use what's known as "Hecke's trick" as it is a clever trick used by Hecke. He considered the series

$$G_1^*(z) = \frac{-1}{8z} \lim_{\epsilon \to 0} \sum_{(m,n)\neq(0,0)} \frac{1}{(nz+m)^2|nz+m|^\epsilon}$$

which converges absolutely, and satisfies the same transformation properties as a weight two modular form (as can be shown by acting by $S$ and $T$ and rearranging the absolutely convergent series). But then it still remains to relate $G_1$ and $G_1^*$, which is done via a Fourier expansion for $G_1^*$ using Poisson summation (a rather robust technique from harmonic analysis that can also be used to obtain the Fourier expansion of $G_k$ for $k \geq 1$.) Instead of employing Poisson summation, we present Apostol's proof, which is somewhat more elementary, but less well motivated:

**Proof** Recopying (2), we have (for $k \geq 1$)

$$\sum_{m\in\mathbb{Z}} \frac{1}{(m+z)^{2k}} = \frac{1}{(2k-1)!}(-2\pi i)^{2k} \sum_{t=1}^{\infty} t^{2k-1} q^t.$$

Note this series converges absolutely for $k \geq 1$ as it is a single sum over integers, not a double sum. Now, with $k = 1$, replace $z$ by $nz$ and sum over positive integers $n$ to obtain:

$$\sum_{n=1}^{\infty} \sum_{m\in\mathbb{Z}} \frac{1}{(m+nz)^2} = (2\pi i)^2 \sum_{n=1}^{\infty}\sum_{t=1}^{\infty} t q^{nt} = (2\pi i)^2 \sum_{\ell=1}^{\infty} \sigma(\ell) q^\ell$$

since the middle sum above is absolutely convergent. Hence,

$$G_2(z) = 2\zeta(2) + \sum_{n\neq 0\in\mathbb{Z}} \sum_{m\in\mathbb{Z}} \frac{1}{(m+nz)^2},$$

but we're not allowed to rearrange this sum.

**Exercise:** Use the above expression to show that

$$z^{-2} G_1\left(\frac{-1}{z}\right) = 2\zeta(2) + \sum_{m\in\mathbb{Z}} \sum_{n\neq 0\in\mathbb{Z}} \frac{1}{(m+nz)^2},$$

that is, the same as above with summation order reversed.

Thus, to prove our claim, we must show that reversing the order of summation has the effect:

$$\sum_{m\in\mathbb{Z}}\sum_{n\neq 0\in\mathbb{Z}}\frac{1}{(m+nz)^2}=\sum_{n\neq 0\in\mathbb{Z}}\sum_{m\in\mathbb{Z}}\frac{1}{(m+nz)^2}-\frac{2\pi i}{z}$$

To prove this, we need an alternate expression of the left-hand side. For this, we can use the Gamma function

$$\Gamma(u)=\int_0^\infty e^{-t}t^{u-1}dt\Rightarrow \alpha^{-u}\Gamma(u)=\int_0^\infty e^{-\alpha t}t^{u-1}dt,$$

for any real $\alpha>0$. This relation can be extended by analytic continuation to any complex $\alpha$ with $\mathrm{Re}(\alpha)>0$. Setting $u=2$ and $\alpha=-2\pi i(m+nz)$ and summing over all $n\geq 1$ gives

$$\sum_{n\neq 0\in\mathbb{Z}}\frac{1}{(nz+m)^2}=-8\pi^2\int_0^\infty\cos(2\pi mu)g_z(u)du$$

where

$$g_t(u)=u\sum_{n=1}^\infty e^{2\pi inzu},\ u>0\qquad g_t(0)=\lim_{u\to 0^+}g_z(u)=\frac{-1}{2\pi iz}.$$

Now we finally arrive at (after summing over $m$ and making a change of variables)

$$\sum_{m\in\mathbb{Z}}\sum_{n\neq 0\in\mathbb{Z}}\frac{1}{(m+nz)^2}=-8\pi^2\sum_{m=-\infty}^\infty\int_0^1 f(t)\cos(2\pi mt)dt$$

where

$$f(t)=\sum_{k=0}^\infty g_z(t+k).$$

The series expression above is a Fourier expansion which converges to the value $1/2(f(0^+)-f(1^-))$, that is, the average approaching 0 from the right and 1 from the left.

**Exercise:** Show that

$$f(0^+)=\frac{-1}{2\pi iz}+\sum_{k=1}^\infty g_z(k)$$

and

$$f(1^-) = \sum_{k=1}^{\infty} g_z(k) = \sum_{n=1}^{\infty} \sigma(n) e^{2\pi i n z}$$

and conclude the theorem. $\square$

**Exercise:** Compute the logarithmic derivative of the product defining $\eta(z)$, relate it to $G_1(z)$, and thus obtain a second proof of the transformation property of $\eta$ under $S$.

# References


[1] J.-P. Serre, *A Course in Arithmetic*, Graduate Texts in Mathematics, vol. 7, Springer-Verlag (1973).