18.781, Fall 2007 Problem Set 11
Due: WEDNESDAY, November 28

These problems all relate to the proof that there are infinitely many primes in an arithmetic progression. While this is covered in Niven, Zuckerman, and Montgomery, our proof differs from theirs and notes are available in the "Solutions and Handouts" section of the course webpage.

1. We will say that an infinite product

$$\prod_{n=1}^{\infty} a_n$$

converges if the limit of the partial products

$$P_N = \prod_{n=1}^{N} a_n$$

converges as $N \to \infty$ and is not 0. (That's the first approximation to the definition. In practice, we want to allow a finite number of the factors a_n in the product to be 0. Suppose N_0 is the largest index with $a_{N_0} = 0$. Then we will say the product converges if

$$P'_N = \prod_{n=N_0+1}^{N} a_n$$

converges as $N \to \infty$.)

   (a) Give an example of an infinite product whose partial products converge to 0. (Hence, we don't regard it as an infinite product.)

   (b) Show that any convergent infinite product can be written in the form

$$\prod_{n=1}^{\infty}(1 + b_n) \quad \text{with} \quad \lim_{n \to \infty} b_n = 0$$

   (c) Part (b) shows that this form for the product terms is a necessary condition for convergence, but give a counterexample to show that it is not sufficient. That is, exhibit an infinite product with terms of the form $(1 + b_n)$ with $\lim_{n \to \infty} b_n = 0$ which does not converge.

   (d) Prove that the Euler product for the Riemann zeta function converges for $s > 1$ (using the above definition). Recall the Euler product has form

$$\zeta(s) = \prod_{p \text{ prime}} \left[1 - p^{-s}\right]^{-1}.$$

2. Attempt to generalize the proof of Dirichlet's theorem on primes in an arithmetic progression to a general modulus d. Our proof works for a prime modulus q. Explain the difficulties that occur in trying to adapt the proof for general moduli. In particular, attempt to define an associated $L(s, \chi)$ to collect primes congruent to $a \bmod d$ for some a with $\gcd(a, d) = 1$. What goes wrong? Can you fix it?

3. In the final stretch of Dirichlet's proof, we must show that

$$L(s, -1) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} \quad \chi(n) = \left(\frac{n}{q}\right)$$

is non-zero at $s = 1$. We know from the Euler product that $L(s) > 0$ for $s > 0$ and so $L(1) \geq 0$. In the case $q \equiv 3\ (4)$, after some manipulation (see notes), we arrive at the expression:

$$L(1) = \frac{-\pi}{q^{3/2}} \sum_{m=1}^{q-1} m \left(\frac{m}{q}\right).$$

Pick several primes $\equiv 3\ (4)$ and calculate this sum. As far as I know, there are no known elementary proofs that the sum above is negative. Attempt to prove this and detail your attempts. (That is, we know that it must be negative, because we know $L(1) \geq 0$ and then we used explicit evaluations of Gauss sums to express $L(1)$ in terms of this sum. Can you see a simpler reason as to why this is true?)