Small Divisors and the NLSE

Bobby Wilson (MSRI)

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Let \( u \in C^k(T) \), \( \int_T u = 0 \), \( \alpha \in \mathbb{R} \), \( T := \mathbb{R}/\mathbb{Z} \).
Find \( v : T \rightarrow \mathbb{R} \) such that

\[
v(x + \alpha) - v(x) = u(x), \text{ for all } x \in T
\]  

(1)
In Fourier coefficients, (1) becomes

\[
(e^{2\pi in\alpha} - 1)\hat{v}(n) = \hat{u}(n), \quad n \in \mathbb{Z} \setminus \{0\}
\]

\[
\hat{v}(n) = (e^{2\pi in\alpha} - 1)^{-1} \hat{u}(n) \approx (\{n\alpha\})^{-1} \hat{u}(n)
\]
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First Problem: \(\alpha\) may be rational.
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\[\hat{v}(n) = (e^{2\pi in\alpha} - 1)^{-1}\hat{u}(n) \approx (\{n\alpha\})^{-1}\hat{u}(n)\]

First Problem: \(\alpha\) may be rational.

Second Problem: For any irrational \(\alpha\) there are \(\infty\)-many rational \(\frac{p}{q}\) such that

\[|\alpha - \frac{p}{q}| < \frac{1}{q^2}\]

Q: Can we get a lower bound for \(|\alpha - \frac{p}{q}|\)?
Definition

α ∉ ℚ is a Diophantine number if ∃c > 0 and r ≥ 2 such that

\[ \left| \alpha - \frac{p}{q} \right| > cq^{-r} \]

for any \( p/q \in \mathbb{Q}, q > 0 \).

Note: Diophantine numbers have full measure.
Definition

$\alpha \notin \mathbb{Q}$ is a Diophantine number if $\exists c > 0$ and $r \geq 2$ such that

$$\left| \alpha - \frac{p}{q} \right| > cq^{-r}$$

for any $p/q \in \mathbb{Q}$, $q > 0$.

Note: Diophantine numbers have full measure.

Assuming $\alpha$ is Diophantine,

$$|\hat{v}(n)| \lesssim_c |n|^{r-1} |\hat{u}(n)|, \quad n \in \mathbb{Z} \setminus \{0\}$$

Consequence: Loss of regularity; $u \in H^k(\mathbb{T}) \Rightarrow v \in H^{k-r+1}(\mathbb{T})$. 
Consider the system

\[ \dot{x} = Ax, \quad A = \begin{pmatrix} i\omega_1 \\ \vdots \\ i\omega_n \end{pmatrix}, \quad \omega_j \in \mathbb{R} \]

If \( \omega = (\omega_1, ..., \omega_n) \) is rationally independent, solutions given by \( x_j(t) = c_j e^{i\omega_j t} \) are quasi-periodic. Does the periodic solution persist under perturbation?
Perturb the system:

\[ \dot{y} = Ay + g(y) \]

where

\[ g(y) = \sum_{|k|_1 \geq 2} g_k y^k, \quad k \in \mathbb{N}^n \setminus \{0\} \]

where \( |k|_1 = k_1 + \cdots + k_n \).
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where \(|k|_1 = k_1 + \cdots + k_n\).

Ansatz for periodic solution:

\[ y(t) = u(e^{At}c), \quad u(x) = x + \sum_{|k|_1 \geq 2} u_k x^k \]

Inserting the ansatz into the perturbed equation, we obtain

\[ \sum_{|k|_1 \geq 2} (\omega \cdot k - A)u_k x^k = g \left( x + \sum_{|k|_1 \geq 2} u_k x^k \right) \]
In coefficients, we have

\[(i \omega \cdot k - A)u_k = \sum_{|m|_1,|\ell|_1 \geq 2, k_i = m_i \ell_i} g_m u_\ell.\]

So we return to the same problem which can be resolved by imposing a similar Diophantine condition:

\[|\omega \cdot k - \omega_j| \geq \frac{\delta}{|k|_1^\tau}\] (2)

for some \(\delta, \tau > 0\). This set of frequency vectors \(\omega\) satisfying (2) has full measure in \(\mathbb{R}^n\).
Statement of a Problem

- Nonlinear Schrödinger Equation

\[ i \partial_t u = \Delta u + \lambda |u|^{2p} u \]  
\[ x \in \mathbb{T}^d, \ \ t \in \mathbb{R}, \ \ p \in \mathbb{N} \]
Nonlinear Schrödinger Equation

\[ i\partial_t u = \Delta u + \lambda |u|^{2p} u \quad (3) \]

\[ x \in \mathbb{T}^d, \quad t \in \mathbb{R}, \quad p \in \mathbb{N} \]

Consider the plane wave solution to (3):

\[ w_m(x, 0) := \varrho e^{im \cdot x} \]

\[ w_m(x, t) = \varrho e^{im \cdot x} e^{i(|m|^2 - \lambda \varrho^2 p) t} \]

Assuming \( u(x, t) \) satisfies (3) and

\[ \|\varrho - e^{-im \cdot x} u(x, 0)\|_{H^s(\mathbb{T}^d)} < \varepsilon, \]

what type of stability can we expect?
solution that starts off near the periodic one

periodic solution

we solve for this difference
Definition (Orbital Stability)

A solution $x(t)$ is said to be orbitally stable if, given $\varepsilon > 0$, there exists a $\delta = \delta(\varepsilon) > 0$ such that, for any other solution, $y(t)$, satisfying $|x(t_0) - y(t_0)| < \delta$, then $d(y(t), O(x_0, t_0)) < \varepsilon$ for $t > t_0$.

- For any $M \in \mathbb{N}$
- There exist $s_0$ and $\varepsilon_0$ so that for any solution $u$ to (3) with $\| \varrho - e^{-im \cdot x} u(x, 0) \|_{H^s(\mathbb{T}^d)} < \varepsilon$, for $\varepsilon < \varepsilon_0$ and $s > s_0$

\[
\inf_{\varphi \in \mathbb{R}} \| e^{-i\varphi} e^{-im \cdot \cdot} w_m(\cdot, t) - e^{-im \cdot \cdot} u(\cdot, t) \|_{H^s(\mathbb{T}^d)} < \varepsilon C(M, s_0, \varepsilon_0)
\]

- For $t < \varepsilon^{-M}$. 

First Approach

- Assume $m = 0$
- Translation of (3) by $w_0$:

$$i \partial_t u = (\Delta + (p + 1)\lambda \varrho^{2p})u + (p\lambda \varrho^{2(p-1)})w_0^2 \bar{u} + \sum_{i=2}^{2p+1} F_i(u, \bar{u}, w_0)$$

(4)

$$i \partial_t u_n = (-|n|^2 + (p + 1)\lambda \varrho^{2p})u_n + (p\lambda \varrho^{2(p-1)})w_0^2 \bar{u}_{-n} + F(u_k, \bar{u}_k, w_0)$$

(5)

- The linear part of (5) is a system with periodic coefficients, so we consider Floquet’s theorem.
Floquet’s Theorem

Theorem (Floquet’s Theorem)

Suppose \( A(t) \) is periodic. Then the Fundamental matrix of the linear system has the form

\[
\Pi(t, t_0) = P(t, t_0) \exp((t - t_0)Q(t_0))
\]

where \( P(\cdot, t_0) \) has the same period as \( A(\cdot) \) and \( P(t_0, t_0) = 1 \).

The eigenvalues of \( M(t_0) := \Pi(t_0 + T, t_0) \), \( \rho_j \), are known as Floquet multipliers and

**Corollary**

A periodic linear system is stable if all Floquet multipliers satisfy

\[ |\rho_j| \leq 1. \]
With \( z_n = e^{-i\lambda \rho^2 p t} u_n \), the linear part of (5) is

\[
i \partial_t \begin{pmatrix} z_n \\ \bar{z}_{-n} \end{pmatrix} = A_n \begin{pmatrix} z_n \\ \bar{z}_{-n} \end{pmatrix}
\]

We then diagonalize

\[
i \partial_t \begin{pmatrix} x_n \\ \bar{x}_{-n} \end{pmatrix} = \begin{pmatrix} \Omega_n & 0 \\ 0 & \Omega_{-n} \end{pmatrix} \begin{pmatrix} x_n \\ \bar{x}_{-n} \end{pmatrix}
\]

where

\[
\Omega_n = \sqrt{|n|^2(|n|^2 + 2p \rho^2 p)}
\]

assuming \( \lambda = -1 \).
Duhamel Iteration Scheme

Duhamel’s Formula:

\[ x_n(t) = e^{i\Omega_n t}x_n(0) + \int_0^t e^{i\Omega_n (t-s)}F(x(s))_n \, ds \]

Define the iteration scheme:

\[
\begin{cases}
  x_n(t, k + 1) = x_n(t, 0) + \int_0^t e^{i\Omega_n (t-s)}F(x_n(s, k)) \, ds \\
  x_n(t, 0) := e^{i\Omega_n t}x_n(0, 0)
\end{cases}
\]
Duhamel Iteration Scheme

Duhamel’s Formula:

\[ x_n(t) = e^{i\Omega_nt}x_n(0) + \int_0^t e^{i\Omega_n(t-s)}F(x(s))_n \, ds \]

Define the iteration scheme:

\[
\begin{align*}
    x_n(t, k + 1) &= x_n(t, 0) + \int_0^t e^{i\Omega_n(t-s)}F(x_n(s, k)) \, ds \\
    x_n(t, 0) &= e^{i\Omega_nt}x_n(0, 0)
\end{align*}
\]

- This approach is similar to the 19th century approach of expanding the solution in a perturbative series:

\[ u(t) = u_0(t) + \varepsilon u_1(t) + \varepsilon^2 u_2(t) + \cdots \]

\(u_k\) being defined recursively.

- This series does not converge, so we should expect a similar phenomenon.
The first step demonstrates issues that this iteration scheme presents us:

Small Model of First Iterate

\[
x_n(t, 1) = x_n(t, 0) + \int_0^t e^{i\Omega_n(t-s)} \sum_{n_1, n_2} x_{n_1}(s, 0)x_{n_2}(s, 0) \, ds \\
= x_n(t, 0) + e^{i\Omega_n t} \sum_{n_1, n_2} x_{n_1}x_{n_2} \int_0^t e^{i(\Omega_{n_1}+\Omega_{n_2}-\Omega_n)s} \, ds \\
= x_n(t, 0) + \sum_{n_1, n_2} x_{n_1}x_{n_2} \frac{e^{i(\Omega_{n_1}+\Omega_{n_2})t} - e^{i\Omega_n t}}{i(\Omega_{n_1} + \Omega_{n_2} - \Omega_n)}
\]
Appearance of small divisors
How do we control the small divisors?

Recall that

$$\Omega_n = \sqrt{|n|^2 (|n|^2 + 2p \varrho^{2p})}$$

and note the pattern

$$\partial_\varrho \Omega_n = \frac{C(n, \varrho)}{\sqrt{|n|^2 + 2p \varrho^{2p}}} = \Omega_n \frac{\tilde{C}(n, \varrho)}{|n|^2 + 2p \varrho^{2p}}$$

$$\partial^2_\varrho \Omega_n = \frac{-C^2(n, \varrho)}{(|n|^2 + 2p \varrho^{2p})^{3/2}} = \Omega_n \frac{-\tilde{C}^2(n, \varrho)}{(|n|^2 + 2p \varrho^{2p})^2}$$

We can conclude that

$$\Omega_{n_1} + \Omega_{n_2} - \Omega_n = \partial_\varrho (\Omega_{n_1} + \Omega_{n_2} - \Omega_n) = \partial^2_\varrho (\Omega_{n_1} + \Omega_{n_2} - \Omega_n) = 0$$

does not occur when \( \varrho \) is restricted to a compact set.
Small Model of Second Iterate

\[ x_n(t, 2) = x_n(t, 0) + \int_0^t e^{i\Omega_n(t-s)} \sum_{n_1, n_2} x_{n_1}(s, 1)x_{n_2}(s, 1) \, ds \]

\[ = x_n(t, 1) + e^{i\Omega_n t} \sum_{n_1, k_1, k_2} \frac{x_{n_1}x_{k_1}x_{k_2} \int_0^t e^{i(\Omega_{n_1} + \Omega_{k_1} + \Omega_{k_2} - \Omega_n)s} - e^{i(\Omega_{n_1} + \Omega_{n_2} - \Omega_n)s} \, ds}{i(\Omega_{k_1} + \Omega_{k_2} - \Omega_{n_2})} \]

\[ + e^{i\Omega_n t} \sum_{j_1, j_2, k_1, k_2} \frac{x_{j_1}x_{j_2}x_{k_1}x_{k_2} \int_0^t e^{i(\Omega_{j_1} + \Omega_{j_2} + \Omega_{k_1} + \Omega_{k_2} - \Omega_n)s} - \ldots \, ds}{-(\Omega_{j_1} + \Omega_{j_2} - \Omega_{n_1})(\Omega_{k_1} + \Omega_{k_2} - \Omega_{n_2})} \]

\[ + \ldots \]
Issues

- Convergence
- Controlling loss of regularity
- Resonances
- Type of stability
  - Problem at zero mode
A Reduction on the Hamiltonian

\[ H := \sum_{k \in \mathbb{Z}^d} |k|^2 |u_k|^2 + \frac{1}{p+1} \sum_{k=1}^{p+1} u_{k_1} \cdots u_{k_{p+1}} \bar{u}_{h_1} \cdots \bar{u}_{h_{p+1}}. \]  

(6)

Let \( L := \|u(0)\|_2^2 \), define the symplectic reduction of \( u_0 \):

\[ \{ u_k, \bar{u}_k \}_{k \in \mathbb{Z}^d} \rightarrow (L, \nu_0, \{ v_k, \bar{v}_k \}_{k \in \mathbb{Z}^d \setminus \{0\}}), \]

\[ u_0 = e^{i\nu_0} \sqrt{L - \sum_{k \in \mathbb{Z}^d} |v_k|^2}, \quad u_k = v_k e^{i\nu_0}, \quad \forall k \in \mathbb{Z}^d \setminus \{0\}. \]
We now diagonalize the quadratic part of the Hamiltonian:

\[ H_0 = \sum_{k \in \mathbb{Z}^d \setminus \{0\}} (k^2 + L^p p)|v_k|^2 + L^p p \frac{1}{2}(v_k v_{-k} + \bar{v}_k \bar{v}_{-k}) \]  

(7)

which gives

\[ H_0 = \sum_{k \in \mathbb{Z}^d} \frac{\Omega_k}{2}(|x_k|^2 + |x_{-k}|^2) \]  

(8)

with \( \Omega_k = \sqrt{|k|^2(|k|^2 + 2pL^p)} \).

- It is convenient to group together the modes having the same frequency, i.e. to denote

\[ \omega_q := \sqrt{q^2(q^2 + 2pL^p)}, \quad q \geq 1. \]  

(9)
Definition (Normal Form)

Let $H = H_0 + P$ where $P \in C^\infty(\mathbb{R}^{2N}, \mathbb{R})$, which is at least cubic such that $P$ is a perturbation of $H_0$. We say that $P$ is in normal form with respect to $H_0$ if it Poisson commutes with $H_0$:

$$\{P, H_0\} = 0$$

Definition (Nonresonance)

Let $r \in \mathbb{N}$. A frequency vector, $\omega \in \mathbb{R}^n$, is nonresonant up to order $r$ if

$$k \cdot \omega := \sum_{j=1}^{n} k_j \omega_j \neq 0 \text{ for all } k \in \mathbb{Z}^n \text{ with } 0 < |k| \leq r$$
Birkhoff Normal Form Theorem in Finite Dimension

**Theorem (Moser '68)**

Let $H = H_0 + P$ where

- $H_0 = \sum_{j=1}^{N} \omega_j \frac{p_j^2 + q_j^2}{2}$
- $P \in C^\infty(\mathbb{R}^{2N}, \mathbb{R})$ having a zero of order 3 at the origin

Fix $M \geq 3$ an integer. There exists $\tau : U \ni (q', p') \mapsto (q, p) \in V$ a real analytic canonical transformation from a nbhd of the origin to a nbhd of the origin which puts $H$ in normal form up to order $M$ i.e.

$$H \circ \tau = H_0 + Z + R$$

with

1. $Z$ is a polynomial of order $r$ and is in normal form
2. $R \in C^\infty(\mathbb{R}^{2N}, \mathbb{R})$ and $R(z, \bar{z}) = O(\|(q, p)\|^{M+1})$
3. $\tau$ is close to the identity: $\tau(q, p) = (q, p) + O(\|(q, p)\|^2)$
Corollary

Assume $\omega$ is nonresonant. For each $M \geq 3$ there exists $\varepsilon_0 > 0$ and $C > 0$ such that if $\|(q_0, p_0)\| = \varepsilon < \varepsilon_0$ the solution $(q(t), p(t))$ of the Hamiltonian system associated to $H$ which takes value $(q_0, p_0)$ at $t = 0$ satisfies

$$\|(q(t), p(t))\| \leq 2\varepsilon \text{ for } |t| \leq \frac{C}{\varepsilon^{M-1}}.$$
Consider the ODE

\[ i \partial_t x_n = \omega_n x_n + \sum_{k \geq 2} \left( f_k(x) \right)_n \]

With

- Auxiliary Hamiltonian: \( \chi(x) \)
- \( X_\chi \) the corresponding vector field

We note that for any vector field \( Y \), its transformed vector field under the time 1 flow generated by \( X_\chi \) is

\[ e^{\text{ad}_{X_\chi}} Y = \sum_{k=0}^{\infty} \frac{1}{k!} \text{ad}_{X_\chi}^k Y \quad (10) \]

where \( \text{ad}_X Y := [Y, X] \).
Iterative Step

- Let \( \chi \) be degree \( K_0 + 1 \)
- Let \( \Phi_\chi(x) \) be the time-1 flow map associated with the Hamiltonian vector field \( X_\chi \).
- Consider the change of variables \( y = \Phi_\chi(x) \)
- Using the identity (10), one obtains

\[
i \partial_t y_n = \omega_n y_n + \sum_{k=2}^{K_0-1} (f_k(y))_n + ([X_\chi, \omega y](y))_n + (f_{K_0}(y))_n + h.o.t.
\]
Plan: choose \( \chi \) and another vector-valued homogeneous polynomial of degree \( K_0, R_{K_0} \), in such a way that we can decompose \( f_{K_0} \) as follows

\[
f_{K_0}(y) = R_{K_0}(y) - [X_\chi, \omega y](y)
\]  

(11)

- We can find \( \chi \) so that \( R_{K_0} \) is in the kernel of the following function

\[
ad_\omega(X) := [X, \omega y].
\]

- Any \( Y \in \ker ad_\omega \) is referred to as "normal" or "resonant".
Condition for a monomial, \( y^{\alpha} \bar{y}^{\beta} \partial_{y^{m}}, (\alpha, \beta \in \mathbb{N}^\infty) \) to satisfy \( y^{\alpha} \bar{y}^{\beta} \partial_{y^{m}} \in \ker \text{ad}_{\omega} \):

\[
\text{ad}_{\omega}(y^{\alpha} \bar{y}^{\beta} \partial_{y^{m}}) = [(\alpha - \beta) \cdot \omega - \omega_{m}]y^{\alpha} \bar{y}^{\beta} \partial_{y^{m}}
\]

For individual terms, (11) becomes

\[
R_{\alpha,\beta,m} - (\omega \cdot (\alpha - \beta) - \omega_{m})X_{\alpha,\beta,m} = f_{\alpha,\beta,m}
\]
Appearance of small divisors

- Condition for a monomial, \( y^\alpha \tilde{y}^\beta \partial y_m \), \((\alpha, \beta \in \mathbb{N}^\infty)\) to satisfy
  \[ y^\alpha \tilde{y}^\beta \partial y_m \in \ker \text{ad}_\omega: \]
  \[ \text{ad}_\omega(y^\alpha \tilde{y}^\beta \partial y_m) = [(\alpha - \beta) \cdot \omega - \omega_m]y^\alpha \tilde{y}^\beta \partial y_m \]

- For individual terms, (11) becomes
  \[ R_{\alpha,\beta,m} - (\omega \cdot (\alpha - \beta) - \omega_m)X_{\alpha,\beta,m} = f_{\alpha,\beta,m} \]

- Definition of \( X_\chi \) and \( R_{K_0} \):
  \[ R_{\alpha,\beta,m} := f_{\alpha,\beta,m} \quad \text{when} \quad \omega \cdot (\alpha - \beta) - \omega_m = 0 \]
  \[ X_{\alpha,\beta,m} := 0 \quad \text{when} \quad \omega \cdot (\alpha - \beta) - \omega_m = 0 \]
  \[ X_{\alpha,\beta,m} := \frac{-f_{\alpha,\beta,m}}{(\omega \cdot (\alpha - \beta) - \omega_m)} \quad \text{when} \quad \omega \cdot (\alpha - \beta) - \omega_m \neq 0 \]
In finite dimension,

$$\inf \{ |\omega \cdot (\alpha - \beta) - \omega_m| \mid \omega \cdot (\alpha - \beta) - \omega_m \neq 0 \} > 0$$

Leads to bound on change-of-variables map (symplectomorphism).

Not necessarily true in infinite dimensions.
Nonresonance Condition

Definition (Nonresonance Condition)

There exists $\gamma = \gamma_M > 0$ and $\tau = \tau_M > 0$ such that for any $N$ large enough, one has

$$\left| \sum_{q \geq 1} \lambda_q \omega_q \right| \geq \frac{\gamma}{N^\tau} \quad \text{for } \|\lambda\|_1 \leq M, \quad \sum_{q > N} |\lambda_q| \leq 2 \quad (12)$$

where $\lambda \in \mathbb{Z}^\infty \setminus \{0\}$. 

The following generalization of the “non-resonance” result in Bambusi-Grebert holds.

Theorem (Bambusi-Grebert 2006)

For any $L_0 > 0$, there exists a set $J \subset (0, L_0)$ of full measure such that if $L \in J$ then for any $M > 0$ the Nonresonance Condition holds.
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There exists $\gamma = \gamma_M > 0$ and $\tau = \tau_M > 0$ such that for any $N$ large enough, one has

$$\left| \sum_{q \geq 1} \lambda_q \omega_q \right| \geq \frac{\gamma}{N^\tau}$$

for $\|\lambda\|_1 \leq M$, $\sum_{q > N} |\lambda_q| \leq 2$ \hspace{1cm} (12)

where $\lambda \in \mathbb{Z}^\infty \setminus \{0\}$.

The following generalization of the “non-resonance” result in Bambusi-Grebert holds.

Theorem (Bambusi-Grebert 2006)

For any $L_0 > 0$, there exists a set $J \subset (0, L_0)$ of full measure such that if $L \in J$ then for any $M > 0$ the Nonresonance Condition holds.
Recall that $\omega_q = \sqrt{q^2(q^2 + 2p\rho^{2p})}$. For any $K \leq N$, consider $K$ indices $j_1 < \cdots < j_K \leq N$. Then

\[
\begin{vmatrix}
\omega_{j_1} & \omega_{j_2} & \cdots & \omega_{j_K} \\
\frac{d}{dm}\omega_{j_1} & \frac{d}{dm}\omega_{j_2} & \cdots & \frac{d}{dm}\omega_{j_K} \\
\cdots & \cdots & \cdots & \cdots \\
\frac{d^{K-1}}{dm^{K-1}}\omega_{j_1} & \frac{d^{K-1}}{dm^{K-1}}\omega_{j_2} & \cdots & \frac{d^{K-1}}{dm^{K-1}}\omega_{j_K}
\end{vmatrix} \gg N^{-2K^2}
\]

where $m = \rho^{2p}$.
Recall that $\omega_q = \sqrt{q^2(q^2 + 2p\rho^{2p})}$. For any $K \leq N$, consider $K$ indices $j_1 < \cdots < j_K \leq N$. Then

$$\left| \begin{array}{ccc}
\omega_{j_1} & \omega_{j_2} & \cdots & \omega_{j_K} \\
\frac{d}{dm}\omega_{j_1} & \frac{d}{dm}\omega_{j_2} & \cdots & \frac{d}{dm}\omega_{j_K} \\
\cdots & \cdots & \cdots & \cdots \\
\frac{d^{K-1}}{dm^{K-1}}\omega_{j_1} & \frac{d^{K-1}}{dm^{K-1}}\omega_{j_2} & \cdots & \frac{d^{K-1}}{dm^{K-1}}\omega_{j_K}
\end{array} \right| \gtrsim N^{-2K^2}$$

where $m = \rho^{2p}$.

Consequently, for $\alpha > 50r^3$. $\forall \gamma > 0$ small enough, $\exists J_\gamma \subset [m_0, m_1]$ such that for all $m \in J_\gamma$, for all $N \geq 1$,

$$\left| \sum_{j=1}^{N} k_j \omega_j + n \right| \geq \frac{\gamma}{N^\alpha}$$

$\forall k \in \mathbb{Z}^N$ with $0 \neq |k| \leq r$ and $\forall n \in \mathbb{Z}$. Moreover,

$$|[m_0, m_1] \setminus J_\gamma| \lesssim \gamma^{1/r}$$
Definition

For $x = \{x_n\}_{n \in \mathbb{Z}^d}$, define the standard Sobolev norm as

$$\| x \|_s := \sqrt{\sum_{n \in \mathbb{Z}^d} |x_n|^2 \langle n \rangle^{2s}}$$

Define $H^s$ as

$$H^s := \{ x = \{x_n\}_{n \in \mathbb{Z}^d} \mid \| x \|_s < \infty \}$$
Consider the equation

\[ i\dot{x} = \omega x + \sum_{k \geq 2} f_k(x). \]  

(13)

and assume the nonresonance condition (12). For any \( M \in \mathbb{N} \), there exists \( s_0 = s_0(M, \tau) \) such that for any \( s \geq s_0 \) there exists \( r_s > 0 \) such that for \( r < r_s \), there exists an analytic canonical change of variables

\[ y = \Phi^{(M)}(x) \]

\[ \Phi^{(M)} : B_s(r) \to B_s(3r) \]

which puts (13) into the normal form

\[ i\dot{y} = \omega y + R^{(M)}(y) + \mathcal{X}^{(M)}(y). \]  

(14)
Moreover there exists a constant $C = C_s$ such that:

\[
\sup_{x \in B_s(r)} \| x - \Phi^{(M)}(x) \|_s \leq Cr^2
\]

- $R^{(M)}$ is at most of degree $M + 2$, is resonant, and has tame modulus
- the following bound holds

\[
\| \mathcal{X}^{(M)} \|_{s,r} \leq Cr^{M+\frac{3}{2}}
\]
Main Theorem: Statement from FGL '13

Theorem (Faou, Gauckler, Lubich 2013)

Let \( \rho_0 > 0 \) be such that \( 1 - 2\lambda \rho_0^2 > 0 \), and let \( M > 1 \) be fixed arbitrarily. There exists \( s_0 > 0 \), \( C \geq 1 \) and a set of full measure \( \mathcal{P} \) in the interval \( (0, \rho_0] \) such that for every \( s \geq s_0 \) and every \( \rho \in \mathcal{P} \), there exists \( \varepsilon_0 \) such that for every \( m \in \mathbb{Z}^d \) the following holds: if the initial data \( u(\bullet, 0) \) are such that

\[
\| u(\bullet, 0) \|_{L^2} = \rho \quad \text{and} \quad \| e^{-im \cdot \bullet} u(\bullet, 0) - u_m(0) \|_{H^s} = \varepsilon \leq \varepsilon_0
\]

then the solution of (3) (with \( p = 1 \)) with these initial data satisfies

\[
\| e^{-im \cdot \bullet} u(\bullet, t) - u_m(t) \|_{H^s} \leq C\varepsilon \quad \text{for} \quad t \leq \varepsilon^{-M}
\]
Structure of the cubic case

Let

\[ H_c = \int_T (|\partial_x u|^2 + |u|^4) \, dx \]

Theorem (Kappeler, Grebert 2014)

There exists a bi-analytic diffeomorphism \( \Omega : H^1 \rightarrow H^1 \) such that \( \Omega \) introduces Birkhoff coordinates for NLS on \( H^1 \). That is, on \( H^1 \) the transformed NLS Hamiltonian \( H_c \circ \Omega^{-1} \) is a real-analytic function of the actions

\[ I_n = \frac{|x_n|^2}{2} \]

for \( n \in \mathbb{Z} \). Furthermore, \( d_0 \Omega \) is the Fourier transform.
Main Theorem: Statement

Theorem (W. 2014)

Let $L_0 > 0$ be such that $1 - 2p\lambda L_0^p > 0$, and let $M > 1$ be fixed arbitrarily. There exists $s_0 > 0$, $C \geq 1$ and a set of full measure $\mathcal{P}$ in the interval $(0, L_0]$ such that for every $s \geq s_0$ and every $L \in \mathcal{P}$, there exists $\varepsilon_0$ such that for every $m \in \mathbb{Z}^d$ the following holds: if the initial data $u(\bullet, 0)$ are such that

$$\|u(\bullet, 0)\|_{L^2}^2 = L \quad \text{and} \quad \|e^{-im \cdot \bullet}u(\bullet, 0) - u_m(0)\|_{H^s} = \varepsilon \leq \varepsilon_0$$

then the solution of (3) with these initial data satisfies

$$\|e^{-im \cdot \bullet}u(\bullet, t) - u_m(t)\|_{H^s} \leq C\varepsilon \quad \text{for} \quad t \leq \varepsilon^{-M}$$
Characterization of $\mathcal{R}^{(M)}$

Proposition

The truncation of (14),

$$i\dot{y} = \omega y + \mathcal{R}^{(M)}(y)$$

can be decoupled in the following way:

$$i\partial_t \begin{pmatrix} y_{n_1} \\ \cdots \\ y_{n_k} \end{pmatrix} = \mathcal{M}_q \begin{pmatrix} y_{n_1} \\ \cdots \\ y_{n_k} \end{pmatrix}$$

(15)

where $q \geq 1$, $\{n_1, \ldots, n_k\} := \{n \in \mathbb{Z}^d : |n| = q\}$, $
\mathcal{M}_q = \mathcal{M}_q(\omega, \{y_j\})$ is a self-adjoint matrix for all $t$. 
Further Questions

- Infinite time result?
- Feasibility of the Floquet/Duhamel iteration
- KAM result
Obstacles

- One parameter family of frequencies
- Repeated frequencies

May be able to overcome this: Bambusi, Berti, Magistrelli

*Degenerate KAM theory for PDEs*


Thank you for listening