The Gauss Map and The 2nd Fundamental Form.

**Def.** Let \( S \subset \mathbb{R}^3 \) be a surface with an orientation \( N \). The map \( N : S \to \mathbb{S}^2 \) takes its values in the unit sphere
\[
\mathbb{S}^2 = \{ (x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1 \}.
\]
The map \( N : S \to \mathbb{S}^2 \), thus defined, is called the Gauss map of \( S \).

The Differential of \( N \)

\( N : S \to \mathbb{S}^2 \) is a smooth map. Thus we can define its differential
\[
dN_p : T_p(S) \to T_{N(p)}(\mathbb{S}^2).
\]

**Claim.** \( T_{N(p)}(\mathbb{S}^2) = T_p(S) \).

It suffices to show that \( T_{N(p)}(\mathbb{S}^2) = \{ v \in \mathbb{R}^3 : \langle v, N(p) \rangle_{\mathbb{R}^3} = 0 \} \subset T_p(S) \).

Let \( \bar{x} : I \to \mathbb{S}^2 \) with \( \bar{x}(0) = N(p) \).

Then
\[
\frac{d}{dt} \| \bar{x}(t) \|^2 \bigg|_{t=0} = \frac{d}{dt} \langle \bar{x}(t), \bar{x}(t) \rangle_{\mathbb{R}^3} \bigg|_{t=0} = 2 \langle \bar{x}(0), \frac{d\bar{x}(0)}{dt} \rangle_{\mathbb{R}^3} \bigg|_{t=0} = 0.
\]

\( \Rightarrow \langle \bar{x}, N(p) \rangle_{\mathbb{R}^3} = 0 \) for all \( v \in T_{N(p)}(\mathbb{S}^2) \).

Thus
\[
dN_p : T_p(S) \to T_p(S).
\]
What about nonorientable surfaces? We define $N, dN_p$ locally.

**Ex.** Consider a plane parametrized by
\[ \mathbf{x}(u,v) = \mathbf{p}_0 + u\mathbf{w}_1 + v\mathbf{w}_2, \quad \|\mathbf{w}_1\| = \|\mathbf{w}_2\| = 1. \]
Then $N(p) = \mathbf{w}_1 \times \mathbf{w}_2$ is constant and $dN_p = 0$.

**Ex.** As before, for $S^2$, we can let.
\[ N(p) = \mathbf{p} = (N_1(p), N_2(p), N_3(p)) = (p_1, p_2, p_3). \]
Let $\gamma: I \to S^2$ then,
\[ \frac{d}{dt} N(\gamma(t)) = \vec{\gamma}'(t). \quad \Rightarrow \quad dN_p \vec{\gamma}'(0) = \vec{\gamma}'(0). \]

**Ex.** Consider the cylinder $T = \{ (x,y,z) \in \mathbb{R}^3 \mid x^2+y^2=1 \}$.
parametrized by $\mathbf{x}(u,v) = (\cos(u), \sin(u), v^3)$.
Then $N = \mathbf{x}_u \times \mathbf{x}_v = (\cos(u), \sin(u), 0)$.
\[ \Rightarrow \quad N(x,y,z) = (x, y, 0). \quad [\text{or } N(x,y,z) = (-x, -y, 0)]. \]
are normal.
\[ \Rightarrow \quad dN_p(\mathbf{v}) = dN_p(v_1, v_2, v_3) = (v_1, v_2, 0). \]
\[ dN_p(\mathbf{w}_1) = \mathbf{w}_2 \quad \text{for } \mathbf{w}_2 \text{ is in xy-plane.} \]
\[ dN_p(\mathbf{w}_2) = 0 \quad \text{for } \mathbf{w}_2 \text{ normal to } xy \text{-plane.} \]
A different look at \( N(p) \cap \text{d}N_p \).

Consider a surface \( S \) and \( p \in S \). Let's try to understand curvature \( \kappa \) at \( p \) by the one-dimensional analogue we know of.

Take \( \tilde{\nu} \in T_p(S) \), \( \| \tilde{\nu} \| = 1 \), \( N(p) \) and intersect \( S \) with the plane through \( p \) spanned by \( \tilde{\nu} \) and \( N(p) \).

Take \( \tilde{\alpha} : [0, 1] \to S \) a plane parametrized by length such that \( \tilde{\alpha}(0) = p \) and \( \tilde{\alpha}'(0) = \tilde{\nu} \).

Since \( \tilde{\alpha} \) is contained in the plane, \( \tilde{N}(0) = \pm N(p) \) and furthermore, \( \tilde{N}(s) = \pm N(\tilde{\alpha}(s)) \).

\[
\tilde{\kappa}(0) = \kappa(0) \quad \tilde{N}(0) \cdot N(p) = \langle \tilde{N}(0), N(p) \rangle_{\mathbb{R}^3} = \langle k(0)\tilde{N}(0), N(p) \rangle_{\mathbb{R}^3} = \langle \tilde{\nu}(0), N(p) \rangle_{\mathbb{R}^3}
\]

\[
\langle N(\tilde{\alpha}(0)), \tilde{\nu} \rangle = -\langle \tilde{\nu}(0), (N \tilde{\alpha})'(0) \rangle_{\mathbb{R}^3} = -\langle \tilde{\nu}(0), dN_p \tilde{\nu} \rangle_{\mathbb{R}^3} = -\langle dN_p \tilde{\nu}, \tilde{\nu} \rangle_{\mathbb{R}^2}
\]

Let's study \( F(\tilde{\nu}, \tilde{\omega}) = -\langle dN_p \tilde{\nu}, \tilde{\omega} \rangle_{T_p(S)} \).

Prop: \( F(\tilde{\nu}, \tilde{\omega}) \) is a symmetric, bilinear form.

\[ F(\tilde{\nu}, \tilde{\omega}) = F(\tilde{\omega}, \tilde{\nu}) \]

Let \( \tilde{x} : U \to S \) be a parametrization of \( S \) near \( p \). It suffices to show that \( F \) is symmetric on the basis \( \{ \tilde{x}_u, \tilde{x}_v \} \).

\[ F(\tilde{x}_u, \tilde{x}_v) = F(\tilde{x}_v, \tilde{x}_u) \]

Note: \( \langle N(p), \tilde{x}_u \rangle_{\mathbb{R}^3} = 0 = \langle N(p), \tilde{x}_v \rangle_{\mathbb{R}^3} \).
Then

$$2 \langle N(p), \overline{x}_u \rangle_{\mathbb{R}^3} = \langle dN_p \overline{x}_u, \overline{x}_u \rangle_{\mathbb{R}^2} + \langle N(p), x_{uv} \rangle_{\mathbb{R}^2}$$

and

$$2 \langle u, N(p), \overline{x}_v \rangle_{\mathbb{R}^3} = \langle dN_p \overline{x}_u, \overline{x}_v \rangle_{\mathbb{R}^2} + \langle N(p), x_{uv} \rangle_{\mathbb{R}^2}$$

$$\langle N(p), x_{uv} \rangle = \langle N(p), \overline{x}_u \rangle \overline{x}_v$$

so

$$F(\overline{x}_u, \overline{x}_v) = - \langle dN_p \overline{x}_u, \overline{x}_v \rangle_{\mathbb{R}^2} = - \langle dN_p \overline{x}_u, \overline{x}_v \rangle_{\mathbb{R}^2} = F(\overline{x}_v, \overline{x}_u).$$

**Def.** The quadratic form \( II_p \), on \( T_p(S) \)
defined by

$$II_p(\overline{v}) = - \langle dN_p \overline{v}, \overline{v} \rangle_{\mathbb{R}^2}$$

$$II_p(\overline{v}, \overline{w}) = - \langle dN_p \overline{v}, \overline{w} \rangle_{\mathbb{R}^2}$$

is called the second fundamental form \( A \) at \( p \).

**Def.** Let \( C \) be a regular curve in \( S \) passing through \( p \) in \( S \), \( k(0) \) the curvature of \( C \) at \( p \), and

$$\cos \theta = \langle N(p), \overline{N}(0) \rangle.$$ 

The number \( k_{N(0)} = k(0) \cos \theta \)

is then called the normal curvature \( k \) at \( C \) at \( S \).

[Diagram: arrow from \( N(p) \) to \( N(0) \).]

**Q:** For a curve \( \overline{c} : (-1, 1) \to S \) with \( \overline{c}(0) = p \)

Is it possible that \( \langle \overline{N}(0), N(p) \rangle = 0 \)?

*Yes, for example, if \( S \) is a plane.*

**Note:** By previous calculation,

$$k_{N(0)} = k(0) \cos \theta = k(0) \langle N(p), \overline{N}(0) \rangle = \langle N(p), \overline{N}(0) \rangle = \langle dN_p \overline{t}(0), \overline{t}(0) \rangle = II_p(\overline{t}(0), \overline{t}(0)).$$
This implies that \( K_n(0) \) doesn't depend on \( \vec{\alpha} \) or the curvature of \( \vec{\alpha} \) at \( t = 0 \).

Why is this surprising?

Many \( \vec{\alpha} \) can have the same tangent, unit normal, and curvature.

Furthermore, for \( \vec{v} \in T_p(S) \), if

\[ \vec{\alpha} : I \to S \cap \text{plane through } p \text{ spanning } \vec{v} \text{ and } N(p) \text{, } \vec{\alpha}(0) = p, \]

then the curvature at \( 0 \) for \( \vec{\alpha} \) is the same as for any other curve satisfying the same conditions.

We can see this by looking at the osculating circle.

**Example:** Let \( \vec{\alpha}, \vec{\beta} : I \to S \) be two regular curves of

\[ \vec{\alpha}(0) = \vec{\beta}(0) = p, \text{ and } (\vec{\alpha}'(0)) = \vec{v} \neq \vec{0} = \vec{\beta}'(0) = \vec{0}, \]

Suppose that the curvatures of \( \vec{\alpha} \) and \( \vec{\beta} \) at \( 0 \) are both 0. The \( II_p(\vec{v}) = 0 \) for all \( \vec{v} \in T_p(S) \),

\[ \det(\vec{v}, II_p(\vec{v})) = 0 \] for all \( \vec{v} \in T_p(S) \).

**Definition:** Let \( p \in S, S \) a regular surface.

Let

\[ K_1 = \max_{\|\vec{v}\| = 1} II_p(\vec{v}) \]

\[ K_2 = \min_{\|\vec{v}\| = 1} II_p(\vec{v}) \]

Then \( K_1 \) and \( K_2 \) are principal curvatures of \( S \) at \( p \).
For every $T_p(S)$, there exists an orthonormal basis $\{e_1, e_2\} \subset T_p(S)$ s.t.

$\text{d}N_p(e_1) = -ke_1$ and $\text{d}N_p(e_2) = -ke_2$

$e_1$ and $e_2$ are called principal directions at $p$.

If a regular connected curve $C$ on $S$ is such that for all $p \in C$ the tangent line of $C$ is a principal direction at $p$, then $C$ is said to be a line of curvature of $S$.

The determinant of $\text{d}N_p : T_p(S) \to T_p(S)$ is the Gaussian curvature $K$ of $S$ at $p$. The negative half of the trace of $\text{d}N_p$ is called the mean curvature $H$ of $S$ at $p$.

$K = k_1k_2 = \det(\text{d}N_p)$

$H = \frac{k_1 + k_2}{2} = \frac{1}{2} \text{tr}(\text{d}N_p)$.

How does one compute the det, trace of a generalized linear operator?

Recall:

$\text{Area}(A_2) = \det(\text{d}N_p) \text{Area}(A_1) \quad \text{for} \quad \frac{A_2}{A_1} = \frac{1}{\sqrt{K \text{Area}(A_1)}}$

This is Gauss's original geometric interpretation.