Ex: Let $S_2$ be an orientable regular surface and $\psi : S_2 \rightarrow S_2$ be a differentiable map, which is a local diffeomorphism at every $p \in S_2$. Prove that $S_1$ is orientable.

It suffices to find $N : S_1 \rightarrow \mathbb{R}^3$ differentiable.

Since $S_2$ is orientable, $\exists N_0 \in \mathbb{R}^3$ differentiable.

Let $p \in S_1$, $\exists U_1, U_2$ open, $U_1 \subset S_2$, $U_2 \subset S_2$ s.t. $p \in U_2 \subset S_2$ and $\psi : U_1 \rightarrow U_2$ is a diffeomorphism.

Then $N = N_0$ is needed.

$$\Rightarrow d(\psi^{-1})_p : T_p(S_2) \rightarrow T_{\psi(p)}(S_2) = T_p(S_1).$$

On $U_2$ $\exists f : U_2 \rightarrow \mathbb{R}$ s.t.

$N_0(q) = f(q)(\bar{x}_u \times \bar{x}_v)$ for some parametrization $\bar{x}$ of $S_2$.

Let $N : U_2 \rightarrow \mathbb{R}^3$, by $N(p) = \left[ \det(d\psi^{-1})_p \right]^{-1} \psi'(q) \left[ \frac{\partial (\psi^{-1})_q \bar{x}_u}{\partial y} \times \frac{\partial (\psi^{-1})_q \bar{x}_v}{\partial y} \right]$.

6. $N(p)$ is normal to $T_p(S_2)$

$d(\psi^{-1})_p$ is an invertible map into $T_p(S_2)$ so $d(\psi^{-1})_q \bar{x}_u$, $d(\psi^{-1})_q \bar{x}_v \in T_p(S_2)$, so the cross product is normal.

- $N(p)$ is a unit vector

$$\|d(\psi^{-1})_q \bar{x}_u \times d(\psi^{-1})_q \bar{x}_v\| = \|\det(d\psi^{-1})_p\| \|\bar{x}_u \times \bar{x}_v\|$$

$$\Rightarrow \|N(p)\| = \|\det(d\psi^{-1})_p\|^{-1} \psi'(q) \left[ \det(d\psi^{-1})_q \|\bar{x}_u \times \bar{x}_v\| = \|N_0(q)\| = 1.\right.$$

- $N(p)$ is differentiable: obvious.
Ex: Show that if a regular surface $S$ contains an open set diffeomorphic to a Möbius strip, then $S$ is non-orientable.

By the previous example, if $S$ was orientable then the Möbius strip would inherit the orientation of $S$ which is impossible.

Prop: If a regular surface is given by

$$S = \{ (x, y, z) \in \mathbb{R}^3 \mid f(x, y, z) = a \},$$

where $f: U \subset \mathbb{R}^3 \to \mathbb{R}$ is differentiable and $a \in \mathbb{R}$ is a regular value of $f$, then $S$ is orientable.

Proof:

Let $\bar{B}: I \to S$ be a differentiable curve on $S$.

Through a point $p \in S$, then

$$f(\bar{B}(t)) = a$$

for all $t \in I$ since $\bar{B}(I) \subset S$.

Thus

$$d_p \bar{B}'(t) = 0$$

for all $t \in I$ since $\bar{B}(I) \subset S$ is arbitrary so $d_p \bar{B}'(t) = 0$.

Thus

$$d_p \bar{B}'(t) = (f_x(p), f_y(p), f_z(p))^T.$$

For all $p \in S$.

Thus

$$N(p) := \left( f_x(p), f_y(p), f_z(p) \right)^T \cdot \frac{1}{\| (f_x(p), f_y(p), f_z(p)) \|^T}$$

is a differentiable field of unit vectors.
Let's consider the converse of the last proposition.

$S$ orientable $\Rightarrow$ exists differentiable $f: U \subset \mathbb{R}^3 \to \mathbb{R}$ s.t. $S = f^{-1}(a)$ and $a$ is a regular value.

Prop. Let $S$ be a regular surface and $\bar{x}: U \to S$ be a parametrization of a nbhd of a point $p = \bar{x}(u_0, v_0) \in S$. Then there exists a nbhd $W \subset \bar{x}(U)$ of $p$ in $S$ and a number $\epsilon > 0$ such that the segments of the normal lines passing through points $q \in W$, with center at $q$ and length $2\epsilon$, are disjoint (that is, $W$ has a tubular nbhd).

Proof: Consider the map $F: U \times \mathbb{R} \to \mathbb{R}^3$ given by

$$F(u, v; t) = \bar{x}(u, v) + t \bar{N}(u, v), \quad (u, v) \in U, \quad t \in \mathbb{R},$$

where $\bar{N}(u, v) = (N_x, N_y, N_z)$ is the unit normal vector at $\bar{x}(u, v)$.

The Jacobian of $F$ at $(u, v; 0)$ is

$$J^{(u, v; t)} = \begin{pmatrix}
(F_1)_u & (F_2)_u & (F_3)_u \\
(F_1)_v & (F_2)_v & (F_3)_v \\
(F_1)_t & (F_2)_t & (F_3)_t
\end{pmatrix} \begin{pmatrix}
\bar{x}_u \\
\bar{x}_v \\
\bar{x}_t
\end{pmatrix}.$$

Thus, $\det(J^{(u, v; 0)}) = \|\bar{x}_u \times \bar{x}_v\| \neq 0.$
By the Inverse Function theorem, there exists $\delta, \varepsilon > 0$

such that $F$ is invertible on the nbhd

$$V = (u_0 - \delta, u_0 + \delta) \times (v_0 - \delta, v_0 + \delta) \times (-\varepsilon, \varepsilon) \subset U \times \mathbb{R}$$

Since $F$ is invertible, $F(\varphi(0,0)) \cap F(\varphi(0,\varepsilon)) = \emptyset$.

Prop: Assume $S \subset \mathbb{R}^3$ is an orientable surface with a tubular nbhd $V \subset \mathbb{R}^3$, and orientation $N : S \to \mathbb{R}^3$.

For each $p \in V$, there exists $\varepsilon > 0$ such that $p = q + \varepsilon N(q)$.

Define $g : V \to \mathbb{R}$ be the map defined by

$$g : p = q + \varepsilon N(q) \mapsto \varepsilon.$$ 

Then $g$ is differentiable and has zero as a regular value.

Consider $F : U \times \mathbb{R} \to \mathbb{R}^3$ from the previous proposition.

and $\varphi : V \to S$ a parametrization.

on $V \subset U \times \mathbb{R}$, $F$ is invertible and

$$F^{-1}(x,y,z) = (u(x,y,z), v(x,y,z), t(x,y,z))$$

is differentiable. Furthermore

$$F^{-1}(p) = F^{-1}(\varphi(u_0, v_0, \varepsilon)) = (u_0, v_0, \varepsilon) = (u_0, v_0, t(p)).$$

The function $t : V \to \mathbb{R}$ is differentiable and $0 \in \mathbb{R}$ is a regular value for $t$.

(Otherwise, $|dF^{-1}| = 0$)

Let $g := t$. 

Compact Surfaces.

Let \( S \subseteq \mathbb{R}^3 \) be a regular, compact, orientable surface. \( \exists \varepsilon > 0 \) s.t. \( \forall p, q \in S \) the segments of the normal lines of length \( 2\varepsilon \), centered at \( p, q \), are disjoint.

Let \( S \) be a compact set. For each \( p \in S \), there exists a nbhd \( U_p \) s.t. \( \exists \varepsilon_p > 0 \) s.t. \( U_p \cap S \) has a tubular nbhd of width \( \varepsilon_p \).

Since \( S \) is compact, we can take a finite subcover \( W_1, \ldots, W_k \) with tubular nbhds of \( U_j \cap S \) of width \( \varepsilon_j > 0 \).

Let \( \varepsilon = \min (\varepsilon_1, \ldots, \varepsilon_k, \varepsilon/2) \) where \( \varepsilon \) is the Lebesgue number of \( W_1, \ldots, W_k \).

Then the \( \varepsilon \)-tubular nbhd of \( S \) is contained in the union of the tubular nbhds over \( W_j \cap S \).

Thus, let \( S \subseteq \mathbb{R}^3 \) be a regular, compact, orientable surface. Then there exists a differentiable function \( g : U \to \mathbb{R} \) defined on the tubular nbhd \( U \) of \( S \), s.t. \( 0 \in \mathbb{R} \) is a regular value of \( g \) and \( S = g^{-1}(0) \).