Surfaces

Def: A subset \( S \subset \mathbb{R}^2 \) is a regular surface if, for each \( p \in S \), there exists a neighborhood \( V \subset \mathbb{R}^3 \) and a map \( \bar{x} : U \to V \) of an open set \( U \subset \mathbb{R}^2 \) onto \( V \) such that:

1. \( \bar{x} \) is differentiable.
   (i.e., \( \bar{x}(u,v) = (x(u,v), y(u,v), z(u,v)) \), \( (u,v) \in U \)
   the functions \( x, y, z \) have continuous partial derivatives of all orders in \( U \).)

2. \( \bar{x} \) is a homeomorphism
   (i.e., \( \exists \bar{x}^{-1} : V \to U \), continuous.

3. (Regularity condition) For each \( q \in U \), the differential
   \( d\bar{x}_q : \mathbb{R}^2 \to \mathbb{R}^3 \)
   is one-to-one (i.e., \( \frac{\partial \bar{x}}{\partial u} \times \frac{\partial \bar{x}}{\partial v} \neq 0 \).

Differential: For \( q \in U \) Let \( \bar{c} \) be a \( \bar{x} \) differentiable curve such that

\( \bar{x}(t_0) = q \)

\( \frac{d}{dt} \bar{x}(\bar{c}(t)) \bigg|_{t=t_0} \)
Example:

Let \( \bar{x} : U \subset \mathbb{R}^2 \rightarrow \mathbb{R}^3 \) be a regular parametrization of the neighborhood \( \mathcal{V} \).

Let \( \bar{x}(u, v) = (x(u, v), y(u, v), z(u, v)) \).

Fix \( q \in U \cap \mathcal{V} \), \( \theta : [-1, 1] \rightarrow \mathbb{R}^2 \) s.t. \( \theta(0) = q \).

Then \( \frac{d}{dt} \bar{x}(\theta(t)) \bigg|_{t=0} = (D_{uv} \bar{x}) \bar{\theta}'(0) \) by chain rule.

\[
= \frac{d}{dt} \bar{x}(\theta(t)) \bigg|_{t=0} = (D_{uv} \bar{x}) \bar{\theta}'(0).
\]

where

\[
D_{uv} \bar{x} \bigg|_{q} = \begin{pmatrix}
\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\
\frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \\
\frac{\partial z}{\partial u} & \frac{\partial z}{\partial v}
\end{pmatrix}
\]

we then let \( d\bar{x}_u := D_{uv} \bar{x} \big|_{q=0} \).

\[\text{Ex.}\]

\( d\bar{x}_u \) is one-to-one is equivalent to \( \frac{\partial x}{\partial u} \times \frac{\partial y}{\partial u} \neq 0 \).

\[\text{Ex.}\]

Show that the unit sphere

\[\mathbb{S}^2 := \{ (x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1 \}\]

is a regular surface.
For $p \in S^2 \cap \{z > 0\}$.

Let $\overline{x}_2(x, y) = (x, y, \sqrt{1-(x^2+y^2)})$.

Here $U = \{(x, y) \mid x^2+y^2 < 1\}$.

Since $x^2+y^2 < 1$, $\sqrt{1-(x^2+y^2)}$ is smooth.

**Condition 2:** The projection $\pi_2(x, y, z) = (x, y)$ is the inverse of $\overline{x}_2$ defined on $U$.

(Why is it continuous?)

**Condition 3:** $\frac{\partial}{\partial x} \overline{x}_2 \cdot \frac{\partial}{\partial y} \overline{x}_2 = \begin{pmatrix} 0 \\ 0 \\ \frac{x}{\sqrt{1-(x^2+y^2)}} \end{pmatrix}$

For $p \in S^2 \cap \{z > 0\}$.

For $p \in S^2 \cap \{z < 0\}$.

Let $\overline{x}_2(x, y) = (x, y, -\sqrt{1-(x^2+y^2)})$.

All conditions are satisfied.

For $p \in S^2 \cap \{z=0\}$, use $\overline{x}_3 = (x, \sqrt{1-(x^2+y^2)}, z)$. We get overlap.
Ex: Show that the Torus, $\mathbb{T}^2$, is a regular surface.

Let $\bar{x}_1(u,v) = ((a + b \cos u) \cos v, (a + b \cos u) \sin v, b \sin u)$

$\bar{x}_2(u,v) = \bar{x}_1(u + \frac{\pi}{2}, v)$

$\bar{x}_3(u,v) = \bar{x}_1(u, v + \frac{\pi}{2})$

$\frac{\partial}{\partial u} \bar{x}_4 = \left( (a-b \sin(u)) \cos v, (a-b \sin(u)) \sin v, b \cos u \right)$

$\frac{\partial}{\partial v} \bar{x}_4 = \left( -(a+b \cos(u)) \sin v, (a+b \cos(u)) \cos v, 0 \right)$

Condition 1:

$\frac{\partial}{\partial u} \bar{x}_4 \times \frac{\partial}{\partial v} \bar{x}_4 = -b(a+b \cos u) \left( \cos u \cos v, \cos u \sin v, \sin u \right) \neq 0$

Condition 2:

Does $\bar{x}_4$ have an inverse?  

Is $\bar{x}_4$ one-to-one?  

If yes, does that imply $\bar{x}_4^{-1}$ is continuous?  

(Note: $f(x) = x^3$ and $f^{-1}(x) = x^{1/3}$)
Prop. Let \( q \in S \) be a pt of a set \( S \subseteq \mathbb{R}^3 \), assume \( f : \mathbb{R}^3 \to \mathbb{R}^3 \) and \( \bar{x} : U \subseteq \mathbb{R}^2 \to V \mathcal{A}S \) be one-to-one, \( C^1 \) differentiable, and \( \frac{\partial}{\partial u} \bar{x} \times \frac{\partial}{\partial v} \bar{x} \neq 0 \) for all \( (u,v) \in U \). Then \( \bar{x}^{-1} \) is continuous.

\[
\frac{\partial}{\partial u} \bar{x} \times \frac{\partial}{\partial v} \bar{x} \neq 0 \rightarrow
\]

either \( |\frac{\partial x}{\partial u} \times \frac{\partial y}{\partial v}| \neq 0 \) or \( |\frac{\partial x}{\partial v} \times \frac{\partial y}{\partial z}| \neq 0 \) or \( |\frac{\partial y}{\partial u} \times \frac{\partial z}{\partial v}| \neq 0 \). Assume WLOG that \( |\frac{\partial x}{\partial v} \times \frac{\partial y}{\partial z}| \neq 0 \).

\( x \) is differentiable at the map \( \pi \circ \bar{x} \) invertible.

By the Inverse Function Theorem, there exist \( U_1, U_2, U_3 \subseteq \mathbb{R}^2 \)

s.t. \( \pi \circ \bar{x}^{-1} : U_1 \to U_2 \) and \( (\pi \circ \bar{x})^{-1} \) is differentiable on \( U_1 \).

Now consider

\[
\begin{align*}
(\pi(u,v), y(u,v), z(u,v)) & \quad \pi^{-1} \quad (\pi \circ \bar{x})^{-1} \\
& \quad \rightarrow (\pi(u,v), y(u,v), z(u,v)) \quad \rightarrow (u,v)
\end{align*}
\]

the inverse of \( \bar{x} \).

Thusly, \( \bar{x}^{-1} = (\pi \circ \bar{x})^{-1} \circ \pi \) is continuous.

(since \( \pi \) and \( (\pi \circ \bar{x})^{-1} \) are continuous), \( \Box \).
Ex. Therefore, $\mathbb{T}^2$ is a regular surface.

**Ex. Stereographic Projection.**

Consider a sphere given by

$$x^2 + y^2 + (z-1)^2 = 1 \quad \text{or} \quad S^2$$

The stereographic projection $\pi_s: S^2 \setminus \{N\} \to \mathbb{R}^2$, which takes the point $(x, y, z) \in S^2$ to the point in the intersection of the $xy$-plane and the line containing $P$ and $N$, where $N = (0, 0, 2)$ is the north pole of $S^2$.

\[\ldots\]

1. One-to-one? Yes, geometrically.

2. Show that $\pi_s^{-1}: \mathbb{R}^2 \to S^2 \setminus \{N\}$.

\[\pi_s^{-1}(u, v) = \left( \frac{4uv}{u^2 + v^2 + 4}, \frac{4v}{u^2 + v^2 + 4}, \frac{2(u^2 + v^2)}{u^2 + v^2 + 4} \right)\]
\( \pi_s^{-1}(u,v) \) is on the line containing \((u,v,0)\) and \((0,0,2)\) and satisfies \(x^2 + y^2 + (z - 1)^2 = 1\).

\[ \Rightarrow \pi_s^{-1}(u,v) = t(u,v,-2) + (0,0,2) \text{ for some } t \in \mathbb{R} \]

and \( t^2 u^2 + t^2 v^2 + (2t+1)^2 = 1 \)

\[ \Rightarrow t^2 (u^2 + v^2 + 4) - 4t = 0 \]

\[ \Rightarrow t = 0 \quad \text{or} \quad t = \frac{4}{u^2 + v^2 + 4} \]

\[ \Rightarrow \pi_s^{-1}(u,v) = \left( \frac{4u}{u^2 + v^2 + 4}, \frac{4v}{u^2 + v^2 + 4}, \frac{2(u^2 + v^2)}{u^2 + v^2 + 4} \right) \]

3) Show that \( \text{span}(\frac{\partial}{\partial u} \pi_s^{-1}, \frac{\partial}{\partial v} \pi_s^{-1}) = \text{span}(\frac{\partial}{\partial x} \bar{x}_2, \frac{\partial}{\partial y} \bar{x}_2) \), where \( \{x^2 + y^2 < 1\} \to S^2 \) is the standard parametrization.

\[ \begin{align*}
\frac{\partial}{\partial x} \bar{x}_2 &= (1, 0, -\sqrt{1-2y^2}/2) \\
\frac{\partial}{\partial y} \bar{x}_2 &= (0, 1, -\sqrt{1-2x^2}/2)
\end{align*} \]

\[ \begin{align*}
\frac{\partial}{\partial u} \pi_s^{-1} &= \left( \frac{4(v^2-u^2+4)}{(u^2+v^2+4)^2}, -\frac{8uv}{(u^2+v^2+4)^2}, \frac{16u}{(u^2+v^2+4)^2} \right) \\
\frac{\partial}{\partial v} \pi_s^{-1} &= \left( -\frac{8uv}{(u^2+v^2+4)^2}, \frac{4(u^2-v^2+4)}{(u^2+v^2+4)^2}, \frac{16v}{(u^2+v^2+4)^2} \right)
\end{align*} \]

\[ \begin{align*}
\frac{\partial}{\partial x} \pi_s^{-1} \times \frac{\partial}{\partial y} \pi_s^{-1} &= \left( \frac{\sqrt{1-4x^2}}{\sqrt{1-4x^2}}, \frac{\sqrt{1-4y^2}}{\sqrt{1-4y^2}} \right)
\end{align*} \]

\[ \begin{align*}
\frac{\partial}{\partial u} \pi_s^{-1} \times \frac{\partial}{\partial v} \pi_s^{-1} &= \frac{1}{(u^2+v^2+4)^3} \begin{pmatrix} -16u & -16v \\ 16 & 16(u^2-v^2) \end{pmatrix}
\end{align*} \]