1) Go through Syllabus

2) Hand Out Index cards

Differential Geometry is the study of local properties of curves and surfaces, and to what extent they give us intrinsic information out curves and surfaces.

Definition: A parametrized, differentiable curve is a differentiable map \( \alpha: I \rightarrow \mathbb{R}^3 \) on an open interval \( I = (a, b) \subset \mathbb{R} \) into \( \mathbb{R}^3 \).

\( \alpha'(t) = (x'(t), y'(t), z'(t)) \) is the tangent/velocity vector of the curve \( \alpha \) at \( t \).

\( \alpha(I) = \{ \alpha(t) \mid t \in I \} \) is the image (called parametrized curve).

**Examples**

First in \( \mathbb{R}^2 \):

\( \alpha(t) = a \cos(t), b \sin(t) \) \( \alpha(t) = (\cos(t), \sin(t)) \)

\( \beta(t) = (\cos(t), \sin(t)) \)

\( \beta(t) = (\cos(t), \sin(t)) \)
2.) \[ \mathbf{x}(t) = (t^2, t^3) \]

\[ \alpha(t) = (t^2-1, t(t^2-1)) \]

3.) \[ \mathbf{y}(t) = (t, |t|) \]

Parametric, smooth curve

4.) Moment Curve

\[ \alpha(t) = (t, t^2, t^3) \]
Example 2 is indicative of the type of issues we run into in Differential Geometry as a whole. When $\mathbf{a}'(t) \neq 0$, we define the line in the direction $\mathbf{a}'(t)$ containing $\mathbf{a}(t)$ which we call the tangent line to $\mathbf{a}$ at $t$.

We will call singular points points $t$ where $\mathbf{a}'(t) = 0$ and when $\mathbf{a}'(t) \neq 0$, it is a regular point.

**Def:** A parametrized, smooth curve $\mathbf{a}$ is regular if $\mathbf{a}'(t) \neq 0$ for all $t \in I$.

**Arc length**

For a curve $\mathbf{a} : [a,b] \rightarrow \mathbb{R}^2$, we take polygonal approximations to estimate length.

$$\text{length } (\mathbf{a}, P) = \sum_{i=1}^{k} \| \mathbf{a}(t_i) - \mathbf{a}(t_{i+1}) \|$$
Then of course a reasonable definition of length is
\[
\text{length } \alpha = \sup_P \| l(\alpha, P) \|_3
\]

**Prop.**
\[
\text{length } \alpha = \frac{1}{2} \int_a^b || \alpha'(t) || \, dt
\]

**Proof.** Let \( s'(t) = \int_a^t || \alpha'(u) || \, du \), length from \( a \) to \( t \).

Fix \( P \),
\[
\text{length } \alpha, P = \sum_{i=1}^n || \alpha(t_{i-1}) - \alpha(t_i) || = \sum_{i=1}^n \int_{t_{i-1}}^{t_i} \| \alpha'(t) \| \, dt \leq \int_a^b || \alpha'(t) || \, dt
\]

\[
\frac{|| \alpha(t+h) - \alpha(t) ||}{h} \leq \frac{\text{length } \alpha(t+h) \text{ to } x.}{h} \leq \frac{|| \alpha'(t) ||}{h}
\]

\[
\Rightarrow s'(t) = || \alpha'(t) || \leq s(b) = \text{length } \alpha = \frac{1}{2} \int_a^b || \alpha'(t) || \, dt
\]

Some examples:
- \( \| \alpha'(t) \| \) constant \( \Rightarrow \alpha, \alpha' \) orthogonal

- Let \( \alpha : [0, 1] \rightarrow \mathbb{R}^2 \) be given by
  \[
  \alpha(t) = (t, t \sin(t))
  \]
  Show that
  \[
  \frac{1}{N} \sum_{n=1}^{N} \| \alpha(t) \| \leq C \sum_{n=1}^{N} \frac{1}{n}
  \]

- Let \( \alpha : [a, b] \rightarrow \mathbb{R}^2 \) is a smooth parametrized plane curve.
  Prove that if \( \| \alpha(s) - \alpha(t) \| \) depends only on \( |s-t| \), then \( \alpha \) must be a subset of a line or circle.

\[
\| \alpha'(t) \| = \lim_{h \to 0} \frac{\| \alpha(t+h) - \alpha(t) \|}{h} = \lim_{h \to 0} \frac{\| \alpha'(t) \|}{h} = || \alpha'(t) || \text{ for all } t
\]

\[
\Rightarrow \alpha'(t) = a(\cos(\sqrt{t}) \), \sin(\sqrt{t}))
\]
\[ \alpha''(t) = a \mathbf{v}'(t) \left( -\sin(\mathbf{v}(t)), \cos(\mathbf{v}(t)) \right) \]

\[ \text{CTOH} \quad \|\alpha''(t)\| = \lim_{h \to 0} \frac{\|\alpha((t+h))-2\alpha(t)+\alpha(t-h)\|}{h^2} \]

Parallelogram Law:
\[ \|\alpha((t+h))-2\alpha(t)+\alpha(t-h)\|^2 = \|\alpha((t+h))-\alpha(t)\|^2 \]
\[ + 2\|\alpha(t)-\alpha(t-h)\|^2 \]
\[ + \|\alpha((t+h))-\alpha(t-h)\|^2 \]
\[ = \|\alpha((t+h))-2\alpha(t)+\alpha(t-h)\|^2 \]

\[ \Rightarrow \|\alpha''(t)\| = \text{constant} \]
\[ \Rightarrow \|\alpha''(t)\| = \|a \mathbf{v}'(t) \left( -\sin(\mathbf{v}(t)), \cos(\mathbf{v}(t)) \right)\| = C \|\mathbf{v}'(t)\| \text{ is constant} \]

Frenet Frame

Assume \( \alpha \) is a parametrization by arc length. Then, i.e. \( \|\alpha'(s)\| = 1 \) for all \( s \).

Let \( \alpha'(s) = \mathbf{T}(s) \)

Then \( \mathbf{T}'(s) \) is orthogonal to \( \mathbf{T}(s) \).

Let \( \mathbf{N}(s) \) be the principal normal vector.

\[ \mathbf{N}(s) := \frac{\mathbf{T}'(s)}{\|\mathbf{T}'(s)\|} \text{ and } \|\mathbf{T}'(s)\| \]

Curvature
\[ K(s) := \|\mathbf{T}'(s)\| \]

(Recall from Calculus)

Define the binormal vector
\[ \mathbf{B}(s) = \mathbf{T}(s) \times \mathbf{N}(s) \]

Then \( \mathbf{T}(s), \mathbf{N}(s), \mathbf{B}(s) \) form a right-handed orthonormal basis for \( \mathbb{R}^3 \).
Some Identities

- \( \vec{N}'(s) \) is a linear combination of \( \vec{T}(s), \vec{N}(s), \vec{B}(s) \),

\[
\vec{N}'(s), \vec{N}(s) = 0 \quad \text{and} \quad \vec{N}'(s) \cdot \vec{T}(s) = -\vec{T}(s) \cdot \vec{N}(s) = -\kappa(s) \vec{T}(s) = -\kappa(s) \vec{T}(s) \Rightarrow \vec{T}'(s) = \kappa(s) \vec{N}(s),
\]

we are left with \( \vec{N}'(s), \vec{B}(s) \) which we let define the torsion

\[
\tau(s) = \vec{N}'(s) \cdot \vec{B}(s),
\]

\[
\Rightarrow \quad \vec{N}'(s) = -\kappa(s) \vec{T}(s) + \tau(s) \vec{B}(s).
\]

- \( \vec{B}'(s), \vec{B}(s) = 0 \), \( \vec{B}'(s), \vec{T}(s) = -\vec{T}(\vec{B}(s)), \vec{B}(s) = 0 \)

and \( \vec{B}'(s), \vec{N}(s) = -\vec{N}'(s), \vec{B}(s) = -\tau(s) \)

\[
\Rightarrow \quad \vec{B}'(s) = -\tau(s) \vec{N}(s).
\]

Frenet Formulas

\[
\begin{align*}
\vec{T}'(s) &= \kappa(s) \vec{N}(s) \\
\vec{N}'(s) &= -\kappa(s) \vec{T}(s) + \tau(s) \vec{B}(s) \\
\vec{B}'(s) &= -\tau(s) \vec{N}(s)
\end{align*}
\]

\[
\begin{bmatrix}
\vec{T}'(s) \\
\vec{N}'(s) \\
\vec{B}'(s)
\end{bmatrix} =
\begin{bmatrix}
\kappa(s) & 0 & -\tau(s) \\
-\kappa(s) & 0 & \tau(s) \\
0 & -\tau(s) & 0
\end{bmatrix}
\begin{bmatrix}
\vec{T}(s) \\
\vec{N}(s) \\
\vec{B}(s)
\end{bmatrix}
\]

Some computations.

Ex: Consider the helix \( \vec{r}(t) = (a \cos t, a \sin t, bt) \).

1) Re-parametrize by length: Find \( s(t) \) s.t. \( \vec{r}'(s(t)) = d(s(t)), \|\vec{r}'(s(t))\| = 1 \)

2) \( \kappa(s) = \frac{a}{c} \)

3) \( \vec{T}(s) = \frac{1}{c} \left( -a \sin \left( \frac{s}{c} \right), a \cos \left( \frac{s}{c} \right), b \right) \)

4) \( \vec{T}'(s) = \frac{a}{c^3} \left( -\cos \left( \frac{s}{c} \right), -\sin \left( \frac{s}{c} \right), 0 \right) \vec{N}(s) \),