Catalan Numbers Page

Content: Below is a list of articles on a diverse topics related to Catalan numbers and their generalizations. I emphasized historically significant works, as well as some bijective, geometric and probabilistic results.

Warning: This list is vastly incomplete as I included only downloadable articles and books (sometimes, by subscription) that I found useful at different times. I do plan to gradually expand it, but will try not to overwhelm the list, so many related results can be obtained by forward and backward reference searches. Let me know if you find it useful.

Basics:

\[ C_n = \frac{1}{n+1} \binom{2n}{n} = \binom{2n}{n} - \binom{2n}{n+1} \text{ for all } n \geq 0. \]

Larger values: at OEIS. Examples and Images: Catalan numbers (MacTutor History of Math.) Another meaning.

Encyclopedia and survey articles:

- OEIS: A108
- English Wikipedia: Catalan number
- MathWorld: Catalan number

http://www.math.ucla.edu/~pak/lectures/Cat/pakcat.htm
Richard Stanley's Exercise 19, solutions, and Catalan addendum (.pdf)
Bonus: the story behind the exercise (the original link)

Catalan numbers history articles:

- Igor Pak, History of Catalan numbers (August 2014), most up to date summary of current historical knowledge; based on historical documents (see below) and two earlier blog posts:
  o "Who computed Catalan numbers?" (Feb 20, 2013).
  o "Who named Catalan numbers?" (Feb 5, 2014).
  o see also Henry Gould's response (May 16, 2014).
- Marc Renault, The Ballot Problem website; original articles and their English translations.
- W. G. Brown, Historical Note on a Recurrent Combinatorial Problem (1965); brief historical timeline and refs.
- P. J. Larcombe and P.D.C. Wilson, On the trail of the Catalan sequence, Math. Today (1998); careful historical overview.
- U. Tamm, Olinde Rodrigues and combinatorics, in Mathematics and social utopias in France (2005); a very readable description of Rodrigues's contributions in the context of history of Catalan numbers.
- J. Laurentin, Réflexions sur la triangulation des polygones convexes, Bulletin de la SABIX (French, 2009); overview of the combinatorial work by Lamé.

Introductory articles:

- M. Hall, Combinatorial theory, Chapter 3 (1967)
- M. Gardner, Catalan Numbers, Scientific American (June 1976)
- P. Hilton and J. Pedersen, Catalan numbers, their generalization, and their uses, Math. Intelligencer (1991)
- D. Rubenstein, Catalan numbers revisited, JCTA (1994)
- M. Aigner, Catalan and other numbers: a recurrent theme, in Algebraic Combinatorics and Computer Science (2001)
- F. Hirzerbuch, FAX to Chern on Catalan numbers (11 November, 2003).

Videos of research lectures:

- Richard Stanley, Some Catalan Musings (see also slides), IMA, Nov 10, 2014.
- Christos Athanasiadis, Catalan and Narayana Numbers for Weyl Groups, MSRI, Nov 5, 2008 (large files).

Historical articles:

- Euler's letter exchange with Goldbach:
**English papers:**

- L. Euler, *Letter to Goldbach* (German, 4 September, 1751). Here are the scans of page 1 and page 2 of the relevant part of the original letter (these are courtesy Xavier Viennot). Euler introduces counting triangulation problem, finds the first 8 Catalan numbers, suggests an explicit product formula, and finds an explicit form g.f.

- C. Goldbach, *Reply to Euler* (German, 16 October, 1751); by Goldbach makes a quick check of the first few terms of Euler's (correct) g.f.

- L. Euler, *Followup letter* (German, 4 December, 1751); Euler shows that the product formula follows from the g.f. and the binomial theorem.

**St. Petersburg papers:**

- J.A. Segner, *Enumeratio modorum quibus figuae planae rectilineae per diagonales dividuntur in triangula* (Size: 13 Mb.), *Novi Commentarii Academiae Scientiarum Imperialis Petropolitanae*, vol. 7 (Latin, 1758/59, published in 1761); proof of a quadratic recurrence relation for Catalan numbers; clearly references Euler's earlier formulation and calculation; uses the formula to compute the first 18 Catalan numbers, the last 6 incorrectly.

- L. Euler, *Unsigned summary* of Segner's article on the number of triangulations, ibid.; here Euler restates his product formula for Catalan numbers, fixes arithmetic mistakes in Segner's calculations, and extends them to the number of triangulations of \( n \)-gon, \( n \leq 25 \). See also my loose English translation.

- S. Kotelnikow, *Demonstratio seriei..., Novi Commentarii*, vol. 10 (Latin, 1766); proves absolutely nothing.

- N. Fuss, *Solutio questionis, quot modis polygonum \( n \) laterum in polyga \( m \) laterum, per diagonales resolvi queat* (Size: 15 Mb.), *Nova Acta Academiae Scientarium Petropolitanae*, vol. 9 (Latin, 1795); Segner recurrence relation for the Fuss-Catalan numbers, thus answering Pfaff's question.

**French (and one German) papers:**

- G. Lamé, *Extract from the letter to J. Liouville on the number of triangulations*, *Jour. de Math.* (French, 1838); a complete elementary combinatorial proof of the recurrence relation and product formula via a double counting argument; English translation by D. Pengelley.


- O. Rodrigues, *Sur le nombre de manières de décomposer un Polygone en triangles au moyen de diagonales*, ibid.; a direct inductive proof of the product formula for the number of triangulations.

- O. Rodrigues, *Sur le nombre de manières d'effectuer un produit de \( n \) facteurs*, ibid.; a direct inductive proof of the product formula for the number of distinct products (bracket sequences).

- M.J. Binet, *Réflexions sur le Problème de déterminer le nombre de manières dont une figure rectiligne peut être partagée en triangles au moyen de ses diagonales*, *Jour. de Math.* (French, 1839); a modern style g.f. proof.

- E. Catalan, *Solution nouvelle de cette question: Un polygone étant donné, de combien de manières peut-on le partager en triangles au moyen de diagonales?,* ibid.; a new recurrence relation.

- J.A. Grunert, *Ueber die Bestimmung der Anzahl der verschiedenen Arten...,* Archive der Mathematik und Physik (German, 1841); product formula for Fuss-Catalan numbers, g.f. approach.

- J. Liouville, *Remarques sur un Mémoire de N. Fuss*, *Jour. de Math.* (French, 1843); a product formula for Fuss-Catalan numbers using Fuss's recurrence and Lagrange inversion.

- l'abbé E. Gelin, *Nombre de manières de décomposer un polygone convexe*, *Mathesis* (1883); a likely case of plagiarism of Lamé's paper; of interest largely due to a historical comment by the editors.

- E. Catalan, *Sur les nombres de Segner*, *Rendiconti del Circolo Matematico di Palermo* (French, 1887); divisibility properties of Catalan numbers.

**French papers:**

- T.P. Kirkman, *On the K-Partitions of the R-Gon and R-Ace*, *Philosophical Transactions of the Royal Society* (1857); introduction of Kirkman-Cayley numbers and a conjectured formula.

- A. Cayley, *On the analytical forms called trees, II*, *Philosophical Magazine* (1859); plane trees, bijection to parenthetical expressions, g.f. solution.

- W. A. Whitworth, *Arrangements of \( m \) things of one sort and \( n \) things of another sort, under certain conditions of priority*, *Messenger of Math* (1879); ballot sequences first defined and computed by a combinatorial argument.

**Papers on the ballot problem:** (translated by M. Renault)
- J. Bertrand, *Solution d’un problème*, *C. R. Acad. Sci.* (French, 1887); *English translation*; ballot problem solution by induction is announced.
- D. André, *Solution directée du problème résolu par M. Bertrand*, ibid.; *English translation*; a different solution, variation on cycle lemma and reflection principle, similar to that by Whitworth.

**Monographs:**
- E. Lucas, *Théorie des nombres* (French, 1891); an early monograph which briefly discussed Catalan numbers after Rodrigues (see *Section 43, Ex IV*).
- E. Netto, *Lehrbuch der Combinatorik* (German, 1901); an early monograph which discussed Catalan numbers after Catalan & Rodrigues, as well as the Schröder numbers (see *Chapter 9*).
- W. Feller, *An Introduction to Probability Theory and its Applications* (see §3.1); this 1950 monograph (mis-)attributed the "reflection principle" to André.

**Trees and other combinatorial interpretations:**

**Polygon dissections and noncrossing partitions:**

**Motzkin, Riordan and Baxter numbers:**
Fine and Schröder numbers:


Lattice walks and generalized ballot numbers:

- T.V. Narayana, *Sur les treilles formés par les partitions d'un entier et leurs applications à la théorie des probabilités*, *C. R. Acad. Sci.* (French, 1955)

Pattern avoidance:


q-Catalan numbers:


q,t-Catalan numbers:


Hyperplane arrangements:

• S. Fishel and M. Vazirani, *A bijection between dominant Shi regions and core partitions*, *European J. Comb.* (2010)

Generalization to Coxeter groups:

• A. Fink and B. Iriarte Giraldo, *A bijection between noncrossing and nonnesting partitions for classical reflection groups*, in *Proc. FPSAC 2009*

Catalan matroid:


Graph of triangulations:


Associahedron:

• A. Tonks, *Relating the associahedron and the permutohedron* (.ps.gz file), in *Operads*, AMS, 1997.
• A. Postnikov, *Permutohedra, associahedra, and beyond*, *IMRN* (2009)
Asymptotic and probabilistic results:


Random walks:


Acknowledgments: The images of road signs and binary trees used above are taken from *Wikipedia*. The first two animated gifs of Dyck paths are takes from *tex.stackexchange.com*, the last one is made by *Alejandro Morales* (it circles over all 5 excited diagrams in a staircase shape).

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