HOMEWORK 9 FOR 18.747, SPRING 2013
DUE FRIDAY, APRIL 19 BY 3PM.

(1) Consider the Airy equation $f'' - zf = 0$. The corresponding system is $f' = g, g' = zf$. Show that the differential Galois group G of this system is $SL_2(\mathbb{C})$.

[Hint. Since the Wronskian of the fundamental system of solutions is 1, this group is contained in $SL_2(\mathbb{C})$. If the group is properly contained in $SL_2(\mathbb{C})$, it is contained in a Borel, and hence has an eigenvector on the space of solutions. Show that this cannot happen.]

(2) It is known that for any invertible matrices $A_1, ..., A_m \in GL_n(\mathbb{C})$ there is a system of differential equations $F' = A(z)F$ on the complex plane with regular singular points at $z_1, ..., z_m$ and infinity whose monodromy representation sends the path around z_i to A_i for all i (in some basis in the solution space). [This is a part of the Riemann-Hilbert correspondence.] Use this to show that any affine algebraic group can be a differential Galois group of a system of linear differential equations on the complex plane with rational functions as coefficients.

[Hint. Show that any affine algebraic group over C has a finitely generated Zariski dense subgroup.]