HOMEWORK 7 FOR 18.747, SPRING 2013
DUE FRIDAY, APRIL 5 BY 3PM.

(1) Let G be a connected linear algebraic group and $B \subset G$ a Borel subgroup.
   (a) Let σ be an automorphism of G, such that $\sigma(g) = g$ for $g \in B$. Show that σ equals identity.
   (b) Let $s : G \to G$ be an onto homomorphism. Show that $s(B') \subset B'$ for some subgroup B' conjugate to B.
   [See Springer's book, pp 112-113 for hints].

(2) Assume $char(k) \neq 2$. Show that SO_n $(n > 2)$ contains semisimple elements whose centralizer is not connected.

(3) Find a solvable subgroup in $SL(2, \mathbb{C})$ which is not conjugate to a subgroup in the group of upper triangular matrices.

(4) (optional) Assume that k is the algebraic closure of a local field, i.e. $k = \mathbb{C}$ or the algebraic closure of $\mathbb{Q}_p$ or $\mathbb{F}_p((t))$.

   [If you are familiar with algebraic groups over non-algebraically closed fields, assume that k is a local field].

   Let $G = GL_n(k)$ [the same can be shown for any reductive group over k]. Equip G with topology induced from the metric topology of $Mat_n(k) \subset k^{n^2}$. We say that an element $g \in G$ contracts an element $x \in G$ is $g^n x g^{-n} \to e$ as $n \to \infty$ and we say that x is g-bounded if the closure of $\{g^n x g^{-n} \mid n \geq 0\}$ is compact.

   Fix $g \in G$, let $U_g \subset G$ be the set of elements contracted by g, and let $P_g \subset G$ be the set of g-bounded elements.
   (a) For any g, the group U_g is the unipotent radical of a parabolic subgroup; in particular, the normalizer of U_g is a parabolic subgroup.
   (b) Show that P_g is contained in the normalizer of U_g; it coincides with that normalizer iff g is semi-simple.
   (c) Explain the relation of this problem to QR-algorithm from numerical linear algebra [see e.g. Strang "Linear algebra"].