HOMEWORK 6 FOR 18.747, SPRING 2013
DUE FRIDAY, MARCH 22 BY 3PM.

(1) Let G be a connected closed subgroup of GL_n. Assume that the space $\mathfrak{g} \subset \mathfrak{gl}_n$ has an $Ad(G)$ invariant complement.

Let $x \in G$ and let C be the conjugacy class of x in GL_n. Show that $C \cap G$ is a finite union of conjugacy classes in G.

[Hint: compare the tangent spaces to $C \cap G$ and the G conjugacy class of x in G.]

(2) Let G be a linear algebraic group and $H \subset G$ be a closed subgroup. Assume that $Hom(H, \mathbb{G}_m)$ is trivial. Show that G/H is *quasi-affine*, i.e. it is an open subvariety of an affine variety.

(3) Assume $char(k) \neq 2$.

  (a) Construct an isomorphism between GL_n/O_n and an open subvariety of symmetric matrices.

  (b) Same question for GL_{2n}/Sp_{2n} and an open subspace of skew-symmetric matrices.

  (c) Compute dimensions of O_n and Sp_{2n}.

  (d) Show that the map $x \mapsto (1+x)^{-1}(1-x)$ exhibits an isomorphism between an open subvariety in SO_n and an open subspace in skew-symmetric matrices. Deduce that SO_n is connected.