HOMEWORK 5 FOR 18.747, SPRING 2013
DUE FRIDAY, MARCH 15 BY 3PM.

(1) Show that if $char(k) = 0$, a homomorphism of algebraic groups over k is an isomorphism iff it is bijective.

(2) Let $\phi : G \to H$ be a surjective homomorphism of algebraic groups of the same dimension. Assume that G is connected. Prove that the kernel of ϕ is contained in the center of G.

(3) In this problem we work over a field k of characteristic 2. Let $H \subset SL_2$ be the centralizer of the matrix u, $u_{11} = u_{12} = u_{22} = 1$, $u_{21} = 0$. Check that the Lie algebra of H does not coincide with the fixed subspace of $Ad(u)$ acting on $\mathfrak{sl}_2$.

(4) Show that for any $g \in GL_n$ the Lie algebra of the centralizer of g coincides with the fixed subspace of $Ad(g)$ acting on $\mathfrak{gl}_n$.

(5) (Optional) This problem relies on some knowledge of elliptic curves and line bundles.

(a) Let E be an elliptic curve. Construct a commutative algebraic group $\tilde{E} \not\cong E \times \mathbb{G}_a$ with an onto homomorphism $\tilde{E} \to E$ whose kernel is isomorphic to $\mathbb{G}_a$.
[Hint: Recall that points of E are in bijection with line bundles on E of degree zero. One can construct $\tilde{E}$ together with a bijection between points of E and line bundles on E of degree zero equipped with a flat connection].

(b) Assume that $char(k) = 0$. Check that $\tilde{E}$ is anti-affine, i.e. any global regular function on E is constant.

(c) Assume that $k = \mathbb{C}$. Construct an analytic group isomorphism $\tilde{E} \cong (\mathbb{C}^\times)^2$.
[Hint: restriction of each character $\tilde{E} \to \mathbb{C}^\times$ to the subgroup $\mathbb{G}_a \subset \tilde{E}$ is a function of the form $z \mapsto \exp(az)$ for some $a \neq 0$. If we think of points of $\tilde{E}$ as line bundles with a flat connection, the homorphism sends such a bundle to monodromy of the connection.]

(d) Assume that $char(k) = p > 0$. Construct an onto homomorphism $\tilde{E} \to \mathbb{G}_a^{(1)}$ whose kernel is identified with $E^{(1)}$; here the upper index ${}^{(1)}$ denotes Frobenius twist.[1]
[Hint: restriction of the character $\tilde{E} \to \mathbb{G}_a$ to the subgroup $\mathbb{G}_a \subset \tilde{E}$ is given by $z \mapsto az^p + bz$ for some a, b with $a \neq 0$. If we think of points of $\tilde{E}$ as line bundles with a flat connection, the homorphism sends such a bundle to p-curvature of the connection.]

[1] Frobenius twist of an algebraic variety X is another algebraic variety which has the same set of points and sheaf of regular functions, but the action of the field k of scalars on the sheaf of functions is twisted by Frobenius automorphism $\lambda \mapsto \lambda^p$. Here k is assumed to be perfect of characteristic $p > 0$.