HOMEWORK 3 FOR 18.747, SPRING 2013
DUE FRIDAY, MARCH 1 BY 3PM.

(1) Let k be a characteristic p field. Let $G \subset GL_{p+1}$ be given by the following equations:
$g_{ij} = 0$ for $i > j$;
$g_{ii} = 1$;
$g_{ij} = \frac{1}{(j-i)!} g_{12}^{(j-i)}$ when $0 < j - i < p$.
Set $x = g_{12}$, $y = g_{1,p+1}$, clearly (x, y) are coordinates on G providing an isomorphism $G \cong \mathbb{A}^2$.
  (a) Check that G is a two dimensional commutative unipotent algebraic group, not isomorphic to the vector group $\mathbb{G}_a^2$.
  (b) Construct an onto homomorphism $G \to \mathbb{G}_a$ whose kernel is isomorphic to $\mathbb{G}_a$.
  (c) (Optional) Consider the map $\pi : G \to G$, $\pi : t \mapsto t^p$. Describe (in coordinates) a homomorphism $V : G \to G$ such that $\pi(x, y) = V(x^p, y^p)$.
  (d) (Optional) For $\lambda \in k$ let T_λ be the endomorphism of G sending (g_{ij}) to $(\lambda^{j-i} g_{ij})$. Describe the subring in the ring $End(G)$ generated by T_λ.
  [This is the ring of *second Witt vectors* over k].
  (e) (Optional) Let $\phi : (x, y) \to (x^p, y^p)$. Show that ϕ, V and T_λ, $\lambda \in k$ generate the ring $End(G)$, and describe this ring explicitly.

(2) Show that any finite subgroup in $\mathbb{G}_a$ is the kernel of a surjective homomorphism $\mathbb{G}_a \to \mathbb{G}_a$.
[Hint: the famous *Artin-Schreier* homomorphism $\mathbb{G}_a \to \mathbb{G}_a$, $t \mapsto t^p - t$ may come handy.]

(3) Assume that k has characteristic p. Find an example of a three dimensional connected unipotent group such that the adjoint action of the group on its Lie algebra is trivial, but the group is non-commutative.
[Hint: Consider the action of $\mathbb{G}_a$ on $\mathbb{G}_a^2$, $t : (x, y) \mapsto (x + t^p y^p, y)$.]

(4) (Optional)
  (a) Show that a two dimensional unipotent group over a field of characteristic zero is commutative.
  (b) Let H be the algebraic group over a field k of characteristic $p > 2$ defined as follows. As an algebraic variety $H \cong \mathbb{A}^2$ with multiplication defined by:
$$(x, z) \cdot (y, w) = \left(x + y, z + w + \frac{1}{2}(x^p y - x y^p) \right).$$
  Show that this is a two-dimensional noncommutative unipotent group.[1]

[1]Some authors call this group a "fake Heisenberg group", see e.g. Boyarchenko, Drinfeld, arXiv:math/0609769, §3.7.