HOMEWORK 2 FOR 18.747, SPRING 2013
DUE FRIDAY, FEBRUARY 22 BY 3PM.

(1) Let G be a unipotent algebraic group and X be an affine G-space. Show that the following are equivalent
  (a) X is a homogeneous space
  (b) Every G-invariant regular function on X is constant.

(2) Fix $n \geq 1$. Show that the statement in the previous problem is no longer true if a unipotent group is replaced by GL_n.

(3) Let $U \subset GL_2$ be the subgroup consisting of matrices (a_{ij}) with $a_{11} = a_{22} = 1$, $a_{21} = 0$. Consider the natural action of U on the space $V = Sym^2(k^2) \cong k^3$ of homogeneous quadratic polynomials in two variables. Describe the orbits of G on V; i.e. give a classification of the orbits and describe each of the orbits by algebraic equations.
  [Hint: the answer depends on whether $char(k) = 2$.]

(4) Assume $char(k) = 0$. Let $C \subset Mat_2(k)$ be the subset of degenerate matrices. Let $G = SL(2)$, the group $G \times G$ acts on the space Mat_2.
  (a) Show that C is a closed G-invariant subvariety, and check that there exists an isomorphism of $G \times G$ modules $O(C) \cong O(G)$.
  [Hint: Let $V = k^2$, consider a map $V \times V^* \to C$, $(v, \xi) \mapsto v \otimes \xi$, and show that it induces an isomorphism between $O(C)$ and the space of bi-homogeneous polynomials on $V \times V^*$ such that the degree in the first variable equals the degree in the second one. Now invoke the description of $O(SL(2))$ presented in class].
  (b) (Optional) Notice that C is a semigroup, the multiplication map $C \times C \to C$ induces a comultiplication on $O(C)$. Show that the isomorphism in part (a) can be chosen to be compatible with comultiplication.
  (c) (Optional) Show that the ring $O(G)$ admits a $G \times G$ invariant increasing filtration, whose associated graded is isomorphic to $O(C)$. Explain how the filtration and the isomorphism interact with comultiplication.
  [Here by an increasing filtration on a ring A we mean a collection of subspaces $A_0 \subset A_1 \subset \dots$, such that $A_i A_j \subset A_{i+j}$ and $\cup A_i = A$. Then associated graded ring is defined by $gr(A) = \oplus(A_i/A_{i-1})$.]