1. Rings, ideals, and modules

1.1. **Rings.** Noncommutative algebra studies properties of rings (not necessarily commutative) and modules over them. By a ring we mean an associative ring with unit 1.

We will see many interesting examples of rings. The most basic example of a ring is the ring $\mathrm{End}M$ of endomorphisms of an abelian group M, or a subring of this ring.

Let us recall some basic definitions concerning rings.

Algebra over a field k: A ring A containing k, such that k is central in A, i.e. $\alpha x = x\alpha$, $\alpha \in k$, $x \in A$.

Invertible element: An element a of a ring A such that there exists $b \in A$ (the inverse of A) for which $ab = ba = 1$.

A (unital) subring: A subset B of a ring A closed under multiplication and containing 1.

Division algebra: A ring A where all nonzero elements are invertible.

Remark 1.1. Let A be a vector space over a field k equipped with a linear map $\mu : A \otimes A \to A$ (the tensor product is over k). Then μ equips A with a structure of a unital associative algebra if and only if $(\mu \otimes Id) \circ \mu = (Id \otimes \mu) \circ \mu$, and there is an element $1 \in A$ such that $\mu(a \otimes 1) = \mu(1 \otimes a) = a$.

Exercises. 1. Show that any division algebra is an algebra over the field of rational numbers $\mathbb{Q}$ or the field $\mathbb{F}_p$ of prime order.

2. Give an example of a ring A and elements $a, b \in A$ such that $ab = 1$ but $ba \neq 1$. Can this happen if A is a finite dimensional algebra over k?

3. Let k be an algebraically closed field, and D a finite dimensional division algebra over k. Show that $D = k$.

4. Let H be a four-dimensional algebra over the field $\mathbb{R}$ of real numbers with basis $1, i, j, k$ and multiplication law $ij = -ji = k, jk = -kj = i, ki = -ik = j, i^2 = j^2 = k^2 = -1$. This algebra is called the algebra of quaternions. Show that H is a division algebra (it is the only noncommutative one over $\mathbb{R}$).

1.2. **Modules.** A **left module** over a ring A is an abelian group M together with a homomorphism of unital rings $\rho : A \to \mathrm{End}M$ (so we require that $\rho(ab) = \rho(a)\rho(b)$, and $\rho(1) = 1$). Alternatively, a left module can be defined as a biadditive map $A \times M \to M$, $(a, v) \mapsto av$, such that $(ab)v = a(bv)$; namely, $av = \rho(a)v$. Modules are also called representations, since in the first version of the definition, we represent each element $a \in A$ by an endomorphism $\rho(a)$.

Remark. Note that if A is an algebra over k then M is a vector space over k, and $\rho(a) : M \to M$ is a linear operator, while the multiplication defines a linear map $A \otimes M \to M$.

A right module can be defined as an abelian group M equipped with a biadditive map $M \times A \to M$, $(a, v) \mapsto va$, such that $v(ab) = (va)b$. It can also be defined as an abelian group M equipped with an antihomomorphism $\rho : A \to \mathrm{End} M$, i.e. $\rho(ab) = \rho(b)\rho(a)$ and $\rho(1) = 1$.

Left (respectively, right) modules over a ring A form a category, where objects are modules and morphisms are module homomorphisms, i.e. group homomorphisms between modules which commute with the action of A. These categories are denoted by $A - \mathrm{Mod}$ and $\mathrm{Mod} - A$, respectively. These categories admit a nice notion of direct sum, as well as those of a submodule, quotient module, kernel and cokernel of a homomorphism; in other words, they are examples of so-called **abelian categories**, which we'll discuss later.

Any ring is automatically a left and right module over itself, via the multiplication map. The same is true for a direct sum of any (not necessarily finite) collection of copies of A. A module of this form is called **free**. It is clear that any module is a quotient of a free module.

A module M is called **irreducible** (or **simple**) if it is nonzero, and its only submodules are 0 and M. A module is called **indecomposable** if it is not isomorphic to a direct sum of two nonzero modules. Clearly, every irreducible module is indecomposable. A module is called **semisimple** if it is isomorphic to a direct sum of simple modules.

Exercises. 5. Give an example of an indecomposable module which is reducible.

6. Let D be a division algebra. Show that any (left or right) D-module M is free (you may assume, for simplicity, that M has finitely many generators). Such modules are called left, respectively right, vector spaces over D.

Remark. The basic theory of vector spaces over a not necessarily commutative division algebra is the same as that in the commutative case, i.e., over a field (if you remember to distinguish between left and right modules), since the commutativity of the field is not used in proofs.

If M is a left A-module, we denote by $\mathrm{End}_A(M)$ the set of module homomorphisms from M to M. This set is actually a ring. It is convenient to define multiplication in this ring by $ab = b \circ a$. This way, M becomes a right module over $\mathrm{End}_A(M)$, i.e., we can write the action of elements of $\mathrm{End}_A(M)$ on M as right multiplications: $m \to mx$, $x \in \mathrm{End}_A(M)$.

Exercise. 7. Show that $\mathrm{End}_A(A) = A$.

1.3. **Ideals.** A subgroup I in a ring A is a **left ideal** if $AI = I$, a **right ideal** if $IA = A$, and a **two-sided ideal** (or simply ideal) if it is both a left and a right ideal. We say that A is a **simple ring** if it does not contain any nontrivial two-sided ideals (i.e., different from 0 and A).

Exercises. 8. If I is a left (respectively, right) ideal in A then $I, A/I$ are left (respectively, right) A-modules (namely, I is a submodule of A). If $I \neq A$ is a two-sided ideal then A/I is a ring, and I is the kernel of the natural homomorphism $A \to A/I$.

9. Let $I_\alpha \subset A$ be a collection of left, right, or two-sided ideals. Then $\sum_\alpha I_\alpha$, $\cap_\alpha I_\alpha$ are left, right, or two-sided ideals, respectively. Also, if I, J are subspaces in A, then the product IJ is a left ideal if so is I and a right

ideal if so is J. In particular, if I is a two-sided ideal, then so is its power I^n.

10. A is a division algebra if and only if any left ideal in A is trivial if and only if any right ideal in A is trivial.

11. Let I be a left (respectively, two-sided) ideal in a ring A. Then the module (respectively, ring) A/I is simple if and only if $I \neq A$ is a maximal ideal in A, i.e. any ideal strictly bigger than I must equal A.

Proposition 1.2. *Let D be a division algebra, and n an integer. The algebra $A := \mathrm{Mat}_n(D)$ is simple.*

Proof. We must show that for any nonzero element $x \in A$, we have $AxA = A$. Let $x = (x_{ij})$, and pick p, q so that $x_{pq} \neq 0$. Then $x_{pq}^{-1} E_{pp} x E_{qq} = E_{pq}$, where E_{ij} are elementary matrices. So $E_{pq} \in AxA$. Then $E_{ij} = E_{ip} E_{pq} E_{qj} \in AxA$, so $AxA = A$. □

Exercise. 12. Check that if D is a division algebra, then D^n is a simple module over $\mathrm{Mat}_n(D)$.

1.4. **Schur's lemma and density theorem.** In this subsection we study simple modules over rings. We start by proving the following essentially trivial statement, which is known as Schur's lemma.

Lemma 1.3. *(i) Let M, N be a simple A-modules, and $f : M \to N$ is a nonzero homomorphism. Then f is an isomorphism.*

(ii) The algebra $\mathrm{End}_A(M)$ of A-endomorphisms of a simple module M is a division algebra.

Proof. (i) The image of f is a nonzero submodule in N, hence must equal N. The kernel of f is a proper submodule of M, hence must equal zero. So f is an isomorphism.

(ii) Follows from (i). □

Schur's lemma allows us to classify submodules in semisimple modules. Namely, let M be a semisimple A-module, $M = \oplus_{i=1}^{k} n_i M_i$, where M_i are simple and pairwise nonisomorphic A-modules, n_i are positive integers, and $n_i M_i$ denotes a direct sum of n_i copies of M_i. Let $D_i = \mathrm{End}_A(M_i)$.

Proposition 1.4. *Let N be a submodule of M. Then N is isomorphic to $\oplus_{i=1}^{k} r_i M_i$, $r_i \leq n_i$, and the inclusion $\phi : N \to M$ is a direct sum of inclusions $\phi_i : r_i M_i \to n_i M_i$, which are given by multiplication of a row vector of elements of M_i (of length r_i) by a certain r_i-by-n_i matrix X_i over D_i with left-linearly independent rows: $\phi_i(m_1, ..., m_{r_i}) = (m_1, ..., m_{r_i}) X_i$. The submodule N coincides with M iff $r_i = n_i$ for all i.*

Proof. The proof is by induction in $n = \sum_{i=1}^{k} n_i$. The base of induction ($n = 1$) is clear. To perform the induction step, let us assume that N is nonzero, and fix a simple submodule $P \subset N$. Such P exists. Indeed, if N itself is not simple, let us pick a direct summand M_s of M such that the projection $p : N \to M_s$ is nonzero, and let K be the kernel of this projection. Then K is a nonzero submodule of $n_1M_1 \oplus ... \oplus (n_s - 1)M_s \oplus ... \oplus n_kM_k$ (as N is not simple), so K contains a simple submodule by the induction assumption.

Now, by Schur's lemma, the inclusion $\phi|_P : P \to M$ lands in n_iM_i for a unique i (such that P is isomorphic to M_i), and upon identification of P with M_i is given by the formula $m \mapsto (mq_1, ..., mq_{n_i})$, where $q_l \in D_i$ are not all zero.

Now note that the group $G_i := GL_{n_i}(D_i)$ of invertible n_i-by-n_i matrices over D_i acts on n_iM_i by right multiplication, and therefore acts on submodules of M, preserving the property we need to establish: namely, under the action of $g \in G_i$, the matrix X_i goes to X_ig, while X_j, $j \neq i$ do not change. Take $g \in G_i$ such that $(q_1, ..., q_{n_i})g = (1, 0, ..., 0)$. Then Ng contains the first summand M_i of n_iM_i, hence $Ng = M_i \oplus N'$, where $N' \subset n_1M_1 \oplus ... \oplus (n_i - 1)M_i \oplus ... \oplus n_kM_k$, and the required statement follows from the induction assumption. The proposition is proved. □

Corollary 1.5. *Let M be a simple A-module, and $v_1, ..., v_n \in M$ be any vectors linearly independent over $D = \mathrm{End}_A(M)$. Then for any $w_1, ..., w_n \in M$ there exists an element $a \in A$ such that $av_i = w_i$.*

Proof. Assume the contrary. Then the image of the map $A \to nM$ given by $a \to (av_1, ..., av_n)$ is a proper submodule, so by Proposition 1.4 it corresponds to an r-by-n matrix X, $r < n$. Let $(q_1, ..., q_n)$ be a vector in D^n such that $X(q_1, ..., q_n)^T = 0$ (it exists due to Gaussian elimination, because $r < n$). Then $a(\sum v_iq_i) = 0$ for all $a \in A$, in particular for $a = 1$, so $\sum v_iq_i = 0$ - contradiction. □

Corollary 1.6. *(the Density Theorem). (i) Let A be a ring and M a simple A-module, which is identified with D^n as a right module over $D = \mathrm{End}_AM$. Then the image of A in $\mathrm{End}M$ is $\mathrm{Mat}_n(D)$.*

(ii) Let $M = M_1 \oplus ... \oplus M_k$, where M_i are simple pairwise nonisomorphic A-modules, identified with $D_i^{n_i}$ as right D_i-modules, where $D_i = \mathrm{End}_A(M_i)$. Then the image of A in $\mathrm{End}M$ is $\oplus_{i=1}^k \mathrm{Mat}_{n_i}(D_i)$.

Proof. (i) Let B be the image of A in $\mathrm{End}M$. Then $B \subset \mathrm{Mat}_n(D)$. We want to show that $B = \mathrm{Mat}_n(D)$. Let $c \in \mathrm{Mat}_n(D)$, and let $v_1, ..., v_n$ be a basis of M over D. Let $w_j = \sum v_ic_{ij}$. By Corollary 1.5, there exists $a \in A$ such that $av_i = w_i$. Then a maps to $c \in \mathrm{Mat}_n(D)$, so $c \in B$, and we are done.

(ii) Let B_i be the image of A in $\mathrm{End}M_i$. Then by Proposition 1.4, $B = \oplus_iB_i$. Thus (ii) follows from (i). □

1.5. **Wedderburn theorem for simple rings.** Are there other simple rings than matrices over a division algebra? Definitely yes.

Exercise. 13. Let A be the algebra of differential operators in one variable, whose coefficients are polynomials over a field k of characteristic zero. That is, a typical element of A is of the form

$$L = a_m(x)\frac{d}{dx}^m + ... + a_0(x),$$

$a_i \in k[x]$. Show that A is simple. (Hint: If I is a nontrivial ideal in A and $L \in I$ a nonzero element, consider $[x, L], [x[x, L]]$,...,to get an nonzero element $P \in I$ which is a polynomial. Then repeatedly commute it with d/dx to show that $1 \in I$, and thus $I = A$). Is the statement true if k has characteristic $p > 0$?

However, the answer becomes "no" if we impose the "descending chain condition". Namely, one says that a ring A satisfies the "descending chain condition" (DCC) for left (or right) ideals if any descending sequence of left (respectively, right) ideals $I_1 \supset I_2 \supset ...$ in A stabilizes, i.e. there is N such that for all $n \geq N$ one has $I_n = I_{n+1}$.

It is clear that if a ring A contains a division algebra D and A is finite dimensional as a left vector space over D, then A satisfies DCC, because any left ideal in A is left vector space over D, and hence the length of any strictly descending chain of left ideals is at most $\dim_D(A)$. In particular, the matrix algebra $\mathrm{Mat}_n(D)$ satisfies DCC. Also, it is easy to show that the direct sum of finitely many algebras satisfying DCC satisfies DCC.

Thus, $\mathrm{Mat}_n(D)$ is a simple ring satisfying DCC for left and right ideals. We will now prove the converse statement, which is known as Wedderburn's theorem.

Theorem 1.7. *Any simple ring satisfying DCC for left or right ideals is isomorphic to a matrix algebra over some division algebra D.*

The proof of the theorem is given in the next subsection.

1.6. **Proof of Wedderburn's theorem.**

Lemma 1.8. *If A is a ring satisfying the DCC for left ideals, and M is a simple A-module, then M is finite dimensional over $D = \mathrm{End}_A(M)$.*

Proof. If M is not finite dimensional, then there is a sequence $v_1, v_2, ...$ of vectors in M which are linearly independent over D. Let I_n be the left ideal in A such that $I_n v_1 = ... = I_n v_n = 0$, then $I_{n+1} \subset I_n$, and $I_{n+1} \neq I_n$, as I_n contains an element a such that $av_{n+1} \neq 0$ (by Corollary 1.5). Thus DCC is violated. □

Now we can prove Theorem 1.7.

Because A satisfies DCC for left ideals, there exists a minimal left ideal M in A, i.e. such that any left ideal strictly contained in M must be zero. Then M is a simple A-module. Therefore, by Schur's lemma, $\mathrm{End}_A M$ is a division algebra; let us denote it by D. Clearly, M is a right module over D. By Lemma 1.8, $M = D^n$ for some n, so, since A is simple, we get that $A \subset \mathrm{Mat}_n(D)$. Since M is simple, by the density theorem $A = \mathrm{Mat}_n(D)$, as desired.

1.7. **The radical and the Wedderburn theorem for semisimple rings.** Let A be a ring satisfying the DCC. The **radical** Rad$A$ of A is the set of all elements $a \in A$ which act by zero in any simple A-module. We have seen above that a simple A-module always exists, thus Rad$A$ is a proper two-sided ideal in A. We say that A is **semisimple** if $\mathrm{RadA} = 0$. So A/RadA is a semisimple ring.

Theorem 1.9. *(Wedderburn theorem for semisimple rings) A ring A satisfying DCC is semisimple if and only if it is a direct sum of simple rings, i.e. a direct sum of matrix algebras over division algebras. Moreover, the sizes of matrices and the division algebras are determined uniquely.*

Proof. Just the "only if" direction requires proof. By the density theorem, it is sufficient to show that A has finitely many pairwise non-isomorphic simple modules. Assume the contrary, i.,e. that $M_1, M_2, ...$ is an infinite sequence of pairwise non-isomorphic simple modules. Then we can define I_m to be the set of $a \in A$ acting by zero in $M_1, ..., M_m$, and by the density theorem the sequence $I_1, I_2, ...$ is a strictly decreasing sequence of ideals, which violates DCC. The theorem is proved. □

In particular, if A is a finite dimensional algebra over an algebraically closed field k, Wedderburn's theorem tells us that A is semisimple if and only if it is the direct sum of matrix algebras $\mathrm{Mat}_{n_i}(k)$. This follows from the fact, mentioned above, that any finite dimensional division algebra over k must coincide with k itself.

Exercise. 14. Show that the radical of a finite dimensional algebra A over an algebraically closed field k is a nilpotent ideal, i.e. some power of it vanishes, and that any nilpotent ideal in A is contained in the radical of A. In particular, the radical of A may be defined as the sum of all its nilpotent ideals.

1.8. **Idempotents and Peirce decomposition.** An element e of an algebra A is called an **idempotent** if $e^2 = e$. If e is an idempotent, so is $1 - e$, and we have a decomposition of A in a direct sum of two modules: $A = Ae \oplus A(1 - e)$. Thus we have

$$A = \mathrm{End}_A(A) = \mathrm{End}_A(Ae) \oplus \mathrm{End}_A(A(1-e)) \oplus \mathrm{Hom}_A(Ae, A(1-e)) \oplus \mathrm{Hom}_A(A(1-e), Ae).$$

It is easy to see that this decomposition can be alternatively written as

$$A = eAe \oplus \oplus (1 - e)A(1 - e) \oplus eA(1 - e) \oplus (1 - e)Ae.$$

This decomposition is called the Peirce decomposition.

More generally, we say that a collection of idempotents $e_1, ..., e_n$ in A is **complete orthogonal** if $e_i e_j = \delta_{ij} e_i$ and $\sum_i e_i = 1$. For instance, for any idempotent e the idempotents $e, 1 - e$ are a complete orthogonal collection. Given such a collection, we have the Pierce decomposition

$$A = \oplus_{i,j=1}^n e_i A e_j,$$

and

$$e_iAe_j = \mathrm{Hom}_A(Ae_i, Ae_j).$$

1.9. **Characters of representations.** We will now discuss some basic notions and results of the theory of finite dimensional representations of an associative algebra A over a field k. For simplicity we will assume that k is algebraically closed.

Let V a finite dimensional representation of A, and $\rho : A \to \mathrm{End}V$ be the corresponding map. Then we can define the linear function $\chi_V : A \to k$ given by $\chi_V(a) = \mathrm{Tr}\rho(a)$. This function is called the character of V.

Let $[A, A]$ denote the span of commutators $[x, y] := xy - yx$ over all $x, y \in A$. Then $[A, A] \subseteq \ker \chi_V$. Thus, we may view the character as a mapping $\chi_V : A/[A, A] \to k$.

Theorem 1.10. *(i) Characters of irreducible finite dimensional representations of A are linearly independent.*

(ii) If A is a finite-dimensional semisimple algebra, then the characters form a basis of $(A/[A, A])^*$.

Proof. (i) If $V_1, \dots, V_r$ are nonisomorphic irreducible finite dimensional representations of A, then

$$\rho_{V_1} \oplus \cdots \oplus \rho_{V_r} : A \to \text{End } V_1 \oplus \cdots \oplus \text{End } V_r$$

is surjective by the density theorem, so $\chi_{V_1}, \dots, \chi_{V_r}$ are linearly independent. (Indeed, if $\sum \lambda_i \chi_{V_i}(a) = 0$ for all $a \in A$, then $\sum \lambda_i \mathrm{Tr}(M_i) = 0$ for all $M_i \in \mathrm{End}_k V_i$. But the traces $\mathrm{Tr}(M_i)$ can take arbitrary values independently, so it must be that $\lambda_1 = \cdots = \lambda_r = 0$.)

(ii) First we prove that $[\mathrm{Mat}_d(k), \mathrm{Mat}_d(k)] = sl_d(k)$, the set of all matrices with trace 0. It is clear that $[\mathrm{Mat}_d(k), \mathrm{Mat}_d(k)] \subseteq sl_d(k)$. If we denote by E_{ij} the matrix with 1 in the ith row of the jth column and 0's everywhere else, we have $[E_{ij}, E_{jm}] = E_{im}$ for $i \neq m$, and $[E_{i,i+1}, E_{i+1,i}] = E_{ii} - E_{i+1,i+1}$. Now, $\{E_{im}\} \cup \{E_{ii} - E_{i+1,i+1}\}$ form a basis in $sl_d(k)$, and thus $[\mathrm{Mat}_d(k), \mathrm{Mat}_d(k)] = sl_d(k)$, as claimed.

By Wedderburn's theorem, we can write $A = \mathrm{Mat}_{d_1}(k) \oplus \cdots \oplus \mathrm{Mat}_{d_r}(k)$. Then $[A, A] = sl_{d_1}(k) \oplus \cdots \oplus sl_{d_r}(k)$, and $A/[A, A] \cong k^r$. By the density theorem, there are exactly r irreducible representations of A (isomorphic to $k^{d_1}, \dots, k^{d_r}$, respectively), and therefore r linearly independent characters in the r-dimensional vector space $A/[A, A]$. Thus, the characters form a basis. $\square$

1.10. **The Jordan-Hölder theorem.** Let A be an associative algebra over an algebraically closed field k. We are going to prove two important theorems about finite dimensional A-modules - the Jordan-Hölder theorem and the Krull-Schmidt theorem.

Let V be a representation of A. A (finite) *filtration* of V is a sequence of subrepresentations $0 = V_0 \subset V_1 \subset \ldots \subset V_n = V$.

Theorem 1.11. *(Jordan-Hölder theorem). Let* V *be a finite dimensional representation of* A*, and* $0 = V_0 \subset V_1 \subset ... \subset V_n = V$*,* $0 = V_0' \subset ... \subset V_m' = V$ *be filtrations of* V*, such that the representations* $W_i := V_i/V_{i-1}$ *and* $W_i' := V_i'/V_{i-1}'$ *are irreducible for all* i*. Then* $n = m$*, and there exists a permutation* σ *of* $1, ..., n$ *such that* $W_{\sigma(i)}$ *is isomorphic to* W_i'*.*

Proof. **First proof** (for k of characteristic zero). Let $I \subset A$ be the annihilating ideal of V (i.e. the set of elements that act by zero in V). Replacing A with A/I, we may assume that A is finite dimensional. The character of V obviously equals the sum of characters of W_i, and also the sum of characters of W_i'. But by Theorem 1.10, the characters of irreducible representations are linearly independent, so the multiplicity of every irreducible representation W of A among W_i and among W_i' are the same. This implies the theorem.

Second proof (general). The proof is by induction on $\dim V$. The base of induction is clear, so let us prove the induction step. If $W_1 = W_1'$ (as subspaces), we are done, since by the induction assumption the theorem holds for V/W_1. So assume $W_1 \neq W_1'$. In this case $W_1 \cap W_1' = 0$ (as W_1, W_1' are irreducible), so we have an embedding $f : W_1 \oplus W_1' \to V$. Let $U = V/(W_1 \oplus W_1')$, and $0 = U_0 \subset U_1 \subset ... \subset U_p = U$ be a filtration of U with simple quotients $Z_i = U_i/U_{i-1}$. Then we see that:

1) V/W_1 has a filtration with successive quotients $W_1', Z_1, ..., Z_p$, and another filtration with successive quotients $W_2, ..., W_n$.

2) V/W_1' has a filtration with successive quotients $W_1, Z_1, ..., Z_p$, and another filtration with successive quotients $W_2', ..., W_m'$.

By the induction assumption, this means that the collection of irreducible modules with multiplicities $W_1, W_1', Z_1, ..., Z_p$ coincides on one hand with $W_1, ..., W_n$, and on the other hand, with $W_1', ..., W_m'$. We are done. ☐

Theorem 1.12. *(Krull-Schmidt theorem) Any finite dimensional representation of* A *can be uniquely (up to order of summands) decomposed into a direct sum of indecomposable representations.*

Proof. It is clear that a decomposition of V into a direct sum of indecomposable representations exists, so we just need to prove uniqueness. We will prove it by induction on $\dim V$. Let $V = V_1 \oplus ... \oplus V_m = V_1' \oplus ... \oplus V_n'$. Let $i_s : V_s \to V$, $i_s' : V_s' \to V$, $p_s : V \to V_s$, $p_s' : V \to V_s'$ be the natural maps associated to these decompositions. Let $\theta_s = p_1 i_s' p_s' i_1 : V_1 \to V_1$. We have $\sum_{s=1}^n \theta_s = 1$. Now we need the following lemma.

Lemma 1.13. *Let* W *be a finite dimensional indecomposable representation of* A*. Then*

(i) Any homomorphism $\theta : W \to W$ *is either an isomorphism or nilpotent;*

(ii) If $\theta_s : W \to W$*,* $s = 1, ..., n$ *are nilpotent homomorphisms, then so is* $\theta := \theta_1 + ... + \theta_n$*.*

Proof. (i) Generalized eigenspaces of θ are subrepresentations of V, and V is their direct sum. Thus, θ can have only one eigenvalue λ. If λ is zero, θ is nilpotent, otherwise it is an isomorphism.

(ii) The proof is by induction in n. The base is clear. To make the induction step ($n-1$ to n), assume that θ is not nilpotent. Then by (i) θ is an isomorphism, so $\sum_{i=1}^{n}\theta^{-1}\theta_i = 1$. The morphisms $\theta^{-1}\theta_i$ are not isomorphisms, so they are nilpotent. Thus $1-\theta^{-1}\theta_n = \theta^{-1}\theta_1+...+\theta^{-1}\theta_{n-1}$ is an isomorphism, which is a contradiction with the induction assumption. $\square$

By the lemma, we find that for some s, θ_s must be an isomorphism; we may assume that $s=1$. In this case, $V_1' = \mathrm{Im} p_1' i_1 \oplus \mathrm{Ker}(p_1 i_1')$, so since V_1' is indecomposable, we get that $f := p_1' i_1 : V_1 \to V_1'$ and $g := p_1 i_1' : V_1' \to V_1$ are isomorphisms.

Let $B = \oplus_{j>1} V_j$, $B' = \oplus_{j>1} V_j'$; then we have $V = V_1 \oplus B = V_1' \oplus B'$. Consider the map $h : B \to B'$ defined as a composition of the natural maps $B \to V \to B'$ attached to these decompositions. We claim that h is an isomorphism. To show this, it suffices to show that $\mathrm{Ker} h = 0$ (as h is a map between spaces of the same dimension). Assume that $v \in \mathrm{Ker} h \subset B$. Then $v \in V_1'$. On the other hand, the projection of v to V_1 is zero, so $gv = 0$. Since g is an isomorphism, we get $v = 0$, as desired.

Now by the induction assumption, $m = n$, and $V_j = V_{\sigma(j)}'$ for some permutation σ of $2, ..., n$. The theorem is proved. $\square$

Exercises. 15. Let A be the algebra of upper triangulaer complex matrices of size n by n.

(i) Decompose A, as a left A-module, into a direct sum of indecomposable modules P_i.

(ii) Find all the simple modules M_j over A, and construct a filtration of the indecomposable modules P_i whose quotients are simple modules.

(iii) Classify finitely generated projective modules over A.

16. Let Q be a quiver, i.e. a finite oriented graph. Let $A(Q)$ be the path algebra of Q over a field k, i.e. the algebra whose basis is formed by paths in Q (compatible with orientations, and including paths of length 0 from a vertex to itself), and multiplication is concatenation of paths (if the paths cannot be concatenated, the product is zero).

(i) Represent the algebra of upper triangular matrices as $A(Q)$.

(ii) Show that $A(Q)$ is finite dimensional iff Q is acyclic, i.e. has no oriented cycles.

(iii) For any acyclic Q, decompose $A(Q)$ (as a left module) in a direct sum of indecomposable modules, and classify the simple $A(Q)$-modules.

(iv) Find a condition on Q under which $A(Q)$ is isomorphic to $A(Q)^{op}$, the algebra $A(Q)$ with opposite multiplication. Use this to give an example of an algebra A that is not isomorphic to A^{op}.

17. Let A be the algebra of smooth real functions on the real line, such that $a(x+1) = a(x)$. Let M be the A-module of smooth functions on the line such that $b(x+1) = -b(x)$.

(i) Show that M is indecomposable and not isomorphic to A, and that $M \oplus M = A \oplus A$ as a left A-module. Thus the conclusion of the Krull-Schmidt theorem does not hold in this case (the theorem fails because the modules we consider are infinite dimensional).

(ii) Classify projective finitely generated A-modules (this is really the classification of real vector bundles on the circle).

18. Let $0 \to X_1 \to X_2 \to X_3$ be a complex in some abelian category (i.e. the composition of any two maps is zero). Show that if for any object Y the corresponding complex $0 \to \mathrm{Hom}(Y, X_1) \to \mathrm{Hom}(Y, X_2) \to \mathrm{Hom}(Y, X_3)$ is exact, then $0 \to X_1 \to X_2 \to X_3$ is exact.

19. **Extensions of representations.** Let A be an algebra over an algebraically closed field k, and V, W be a pair of representations of A. We would like to classify representations U of A such that V is a subrepresentation of U, and $U/V = W$. Of course, there is an obvious example $U = V \oplus W$, but are there any others?

Suppose we have a representation U as above. As a vector space, it can be (non-uniquely) identified with $V \oplus W$, so that for any $a \in A$ the corresponding operator $\rho_U(a)$ has block triangular form

$$\rho_U(a) = \begin{pmatrix} \rho_V(a) & f(a) \\ 0 & \rho_W(a) \end{pmatrix},$$

where $f : A \to \mathrm{Hom}_k(W, V)$.

(a) What is the necessary and sufficient condition on $f(a)$ under which $\rho_U(a)$ is a representation? Maps f satisfying this condition are called (1-)cocycles (of A with coefficients in $\mathrm{Hom}_k(W, V)$). They form a vector space denoted $Z^1(W, V)$.

(b) Let $X : W \to V$ be a linear map. The coboundary of X, dX, is defined to be the function $A \to \mathrm{Hom}_k(W, V)$ given by $dX(a) = \rho_V(a)X - X\rho_W(a)$. Show that dX is a cocycle, which vanishes iff X is a homomorphism of representations. Thus coboundaries form a subspace $B^1(W, V) \subset Z^1(W, V)$, which is isomorphic to $\mathrm{Hom}_k(W, V)/\mathrm{Hom}_A(W, V)$. The quotient $Z^1(W, V)/B^1(W, V)$ is denoted $\mathrm{Ext}^1(W, V)$.

(c) Show that $f, f' \in Z^1(W, V)$ and $f - f' \in B^1(W, V)$ then the corresponding extensions U, U' are isomorphic representations of A. Conversely, if $\phi : U \to U'$ is an isomorphism such that

$$\phi(a) = \begin{pmatrix} 1_V & * \\ 0 & 1_W \end{pmatrix}$$

then $f - f' \in B^1(V, W)$. Thus, the space $\mathrm{Ext}^1(W, V)$ "classifies" extensions of W by V.

(d) Assume that W, V are finite dimensional irreducible representations of A. For any $f \in Ext^1(W, V)$, let U_f be the corresponding extension. Show that U_f is isomorphic to $U_{f'}$ as representations if and only if f and f' are proportional. Thus isomorphism classes (as representations) of nontrivial extensions of W by V (i.e., those not isomorphic to $W \oplus V$) are parametrized by the projective space $\mathbb{P}\mathrm{Ext}^1(W,V)$. In particular, every extension is trivial iff $\mathrm{Ext}^1(W,V)=0$.

20. (a) Let $A=\mathbf{C}[x_1,...,x_n]$, and V_a, V_b be one-dimensional representations in which x_i act by a_i and b_i, respectively ($a_i, b_i \in \mathbf{C}$). Find $\mathrm{Ext}^1(V_a,V_b)$ and classify 2-dimensional representations of A.

(b) Let B be the algebra over $\mathbf{C}$ generated by $x_1,...,x_n$ with the defining relations $x_ix_j=0$ for all i,j. Show that for $n>1$ the algebra B has only one irreducible representation, but infinitely many non-isomorphic indecomposable representations.

21. Let Q be a quiver without oriented cycles, and P_Q the path algebra of Q. Find irreducible representations of P_Q and compute Ext^1 between them. Classify 2-dimensional representations of P_Q.

22. Let A be an algebra, and V a representation of A. Let $\rho: A\to \mathrm{End}V$. A formal deformation of V is a formal series

$$\tilde{\rho}=\rho_0+t\rho_1+...+t^n\rho_n+...,$$

where $\rho_i: A\to \mathrm{End}(V)$ are linear maps, $\rho_0=\rho$, and $\tilde{\rho}(ab)=\tilde{\rho}(a)\tilde{\rho}(b)$.

If $b(t)=1+b_1t+b_2t^2+...$, where $b_i\in\mathrm{End}(V)$, and $\tilde{\rho}$ is a formal deformation of ρ, then $b\tilde{\rho}b^{-1}$ is also a deformation of ρ, which is said to be isomorphic to $\tilde{\rho}$.

(a) Show that if $\mathrm{Ext}^1(V,V)=0$, then any deformation of ρ is trivial, i.e. isomorphic to ρ.

(b) Is the converse to (a) true? (consider the algebra of dual numbers $A=k[x]/x^2$).

23. Let A be the algebra over complex numbers generated by elements g,x with defining relations $gx=-xg, x^2=0, g^2=1$. Find the simple modules, the indecomposable projective modules, and the Cartan matrix of A.

24. We say that a finite dimensional algebra A has homological dimension d if every finite dimensional A-module M admits a projective resolution of length d, i.e. there exists an exact sequence $P_d\to P_{d-1}\to ...\to P_0\to M\to 0$, where P_i are finite dimensional projective modules. Otherwise one says that A has infinite homological dimension.

(a) Show that if A has finite homological dimension d, and C is the Cartan matrix of A, then $\det(C)=\pm 1$.

(b) What is the homological dimension of $k[t]/t^n$, $n>1$? Of the algebra of problem 23?

25. Let Q be a finite oriented graph without oriented cycles.

(a) Find the Cartan matrix of its path algebra $A(Q)$.

(b) Show that $A(Q)$ has homological dimension 1.

26. Let $\mathcal{C}$ be the category of modules over a k-algebra A. Let F be the forgetful functor from this category to the category of vector spaces, and Id the identify functor of $\mathcal{C}$.

(a) Show that the algebra of endomorphisms of F is naturally isomorphic to A.

(b) Show that the algebra of endomorphisms of Id is naturally isomorphic to the center $Z(A)$ of A.

27. **Blocks.** Let A be a finite dimensional algebra over an algebraically closed field k, and $\mathcal{C}$ denote the category of finite dimensional A-modules. Two simple finite dimensional A-modules X, Y are said to be linked if there is a chain $X = M_0, M_1, ..., M_n = Y$ such that for each $i = 1, ..., n$ either $\mathrm{Ext}^1(M_i, M_{i+1}) \neq 0$ or $\mathrm{Ext}^1(M_{i+1}, M_i) \neq 0$ (or both). This linking relation is clearly an equivalence relation, so it defines a splitting of the set S of simple A-modules into equivalence classes S_k, $k \in B$. The k-th block $\mathcal{C}_k$ of $\mathcal{C}$ is, by definition, the category of all objects M of $\mathcal{C}$ such that all simple modules occuring in the Jordan-Hölder series of M are in S_k.

(a) Show that there is a natural bijection between blocks of $\mathcal{C}$ and indecomposable central idempotents e_k of A (i.e. ones that cannot be nontrivially split in a sum of two central idempotents), such that $\mathcal{C}_k$ is the category of finite dimensional $e_k A$-modules.

(b) Show that any indecomposable object of $\mathcal{C}$ lies in some $\mathcal{C}_k$, and $\mathrm{Hom}(M, N) = 0$ if $M \in \mathcal{C}_k$, $N \in \mathcal{C}_l$, $k \neq l$. Thus, $\mathcal{C} = \oplus_{k \in B} \mathcal{C}_k$.

28. Let A be a finitely generated algebra over a field k. One says that A has polynomial growth if there exists a finite dimensional subspace $V \subset A$ which generates A, and satisfies the "polynomial growth condition": there exist $C > 0$, $k \geq 0$ such that one has $\dim(V^n) \leq Cn^k$ for all $n \geq 1$ (where $V^n \subset A$ is the span of elements of the form $a_1 ... a_n$, $a_i \in V$).

(a) Show that if A has polynomial growth then the polynomial growth condition holds for *any* finite dimensional subspace of A.

(b) Show that if V is a finite dimensional subspace generating A, and $[V, V] \subset V$ (where $[V, V]$ is spanned by $ab - ba$, $a, b \in V$) then A has polynomial growth. Deduce that the algebra D_n of differential operators with polynomial coefficients in n variables and the universal enveloping algebra $U(\mathfrak{g})$ of a finite dimensional Lie algebra $\mathfrak{g}$ have polynomial growth.

(c) Show that the algebra generated by x, y with relation $xy = qyx$ (the q-plane) has polynomial growth ($q \in k^\times$).

(d) Recall that a nilpotent group is a group G for which the lower central series $L_1(G) = G$, $L_{i+1}(G) = [G, L_i(G)]$ degenerates, i.e., $L_n(G) = \{1\}$ for some n (here $[G, L_i(G)]$ is the group generated by $aba^{-1}b^{-1}$, $a \in G$, $b \in L_i(G)$). Let G be a finitely generated nilpotent group. Show that the group algebra $k[G]$ has polynomial growth (the group algebra has basis $g \in G$ with multiplication law $g * h := gh$).

29. Show that if A is a domain (no zero divisors) and has polynomial growth, then the set $S = A \setminus 0$ of nonzero elements of A is a left and right Ore set, and AS^{-1} is a division algebra (called the skew field of quotients of A). Deduce that the algebras $D_n, U(\mathfrak{g})$, the q-plane have skew fields of quotients. Under which condition on the nilpotent group G is it true for $k[G]$?

30. (a) Show that any ring has a maximal left (and right) ideal (use Zorn's lemma).

(b) We say that a module M over a ring A has splitting property if any submodule N of M has a complement Q (i.e., $M = N \oplus Q$). Show that M has splitting property if and only if it is semisimple, i.e. a (not necessarily finite) direct sum of simple modules.

Hint. For the "only if" direction, show first that a module with a splitting property has a simple submodule (note that this is NOT true for an arbitrary module, e.g. look at $A = k[t]$ regarded as an A-module!). For this, consider a submodule N of M generated by one element, and show that N is a quotient of M, and that N has a simple quotient S (use (a)). Conclude that S is a simple submodule of M. Then consider a maximal semisimple submodule of M (use Zorn's lemma to show it exists).

31. **Hochschild homology and cohomology.** Let A be an associative algebra over a field k. Consider the complex $C^\bullet(A)$ defined by $C^i(A) = A^{\otimes i+2}$, $i \geq -1$, with the differential $d : C^i(A) \to C^{i-1}(A)$ given by the formula

$$d(a_0 \otimes a_1 ... \otimes a_{i+1}) = a_0a_1 \otimes ... \otimes a_{i+1} - a_0 \otimes a_1a_2 \otimes ... \otimes a_{i+1} ... + (-1)^{i-1} a_0 \otimes ... \otimes a_ia_{i+1}.$$

(a) Show that $(C^\bullet(A), d)$ is a resolution of A by free A-bimodules (i.e. right $A^\circ \otimes A$-modules), i.e. it is an exact sequence, and $C^i(A)$ are free for $i \geq 0$.

(b) Use this resolution to write down explicit complexes to compute the spaces $\mathrm{Ext}^i_{A^\circ \otimes A}(A, M)$ and $\mathrm{Tor}_i^{A^\circ \otimes A}(A, M)$, for a given A-bimodule M. These spaces are called the Hochschild cohomology and homology spaces of A with coefficients in M, respectively, and denoted $HH^i(A, M)$ and $HH_i(A, M)$.

(c) Show that $HH^0(A, A)$ is the center of A, $HH_0(A, A) = A/[A, A]$, $HH^1(A, A)$ is the space of derivations of A modulo inner derivations (i.e. commutators with an element of A).

(d) Let A_0 be an algebra over a field k. An n-th order deformation of A_0 is an associative algebra A over $k[t]/t^{n+1}$, free as a module over $k[t]/t^{n+1}$, together with an isomorphism of k-algebras $f : A/tA \to A_0$. Two such deformations (A, f) and (A', f') are said to be equivalent if there exists an algebra isomorphism $g : A \to A'$ such that $f'g = f$. Show that equivalence classes of first order deformations are parametrized by $HH^2(A_0, A_0)$.

(e) Show that if $HH^3(A_0, A_0) = 0$ then any n-th order deformation can be lifted to (i.e., is a quotient by t^{n+1} of) an $n + 1$-th order deformation.

(f) Compute the Hochschild cohomology of the polynomial algebra $k[x]$. (Hint: construct a free resolution of length 2 of $k[x]$ as a bimodule over itself).

32. (a) Prove the Künneth formula:

If A, B have resolutions by finitely generated free bimodules, then

$$HH^i(A \otimes B, M \otimes N) = \oplus_{j+k=i} HH^j(A, M) \otimes HH^k(B, N).$$

(b) Compute the Hochschild cohomology of $k[x_1, ..., x_m]$.

33. Let k be a field of characteristic zero.

(a) Show that if V is a finite dimensional vector space over k, and $A_0 = k[V]$, then $HH^i(A_0, A_0)$ is naturally isomorphic to the space of polyvector fields on V of rank i, $k[V] \otimes \wedge^i V$, i.e. the isomorphism commutes with $GL(V)$ (use 32(b)).

(b) According to (a), a first order deformation of A_0 is determined by a bivector field $\alpha \in k[V] \otimes \wedge^2 V$. This bivector field defines a skew-symmetric bilinear binary operation on $k[V]$, given by $\{f, g\} = (df \otimes dg)(\alpha)$. Show that the first order deformation defined by α lifts to a second order deformation if and only if this operation is a Lie bracket (satisfies the Jacobi identity). In this case α is said to be a Poisson bracket.

Remark. A deep theorem of Kontsevich says that if α is a Poisson bracket then the deformation lifts not only to the second order, but actually to all orders. Curiously, all known proofs of this theorem use analysis, and a purely algebraic proof is unknown.

(c) Give an example of a first order deformation not liftable to second order.

34. Let A be an n-th order deformation of an algebra A_0, and M_0 be an A_0-module. By an m-th order deformation of M_0 (for $m \le n$) we mean a module M_m over $A_m = A/t^{m+1}A$, free over $k[t]/(t^{m+1})$, together with an identification of M_m/tM_m with M_0 as A_0-modules.

(a) Assume that $n \ge 1$. Show that a first order deformation of M_0 exists iff the image of the deformation class $\gamma \in HH^2(A_0, A_0)$ of A under the natural map $HH^2(A_0, A_0) \to HH^2(A_0, \mathrm{End} M_0) = \mathrm{Ext}^2(M_0, M_0)$ is zero.

(b) Show that once one such first order deformation ξ is fixed, all the first order deformations of M_0 are parametrized by elements $\beta \in HH^1(A_0, \mathrm{End} M_0) = \mathrm{Ext}^1(M_0, M_0)$.

(c) Show that if $\mathrm{Ext}^2(M_0, M_0) = 0$ then any first order deformation of M_0 is liftable to n-th order.

35. Show that any finite dimensional division algebra over the field $k = \mathbb{C}((t))$ is commutative.

Hint. Start with showing that any finite extension of k is $\mathbb{C}((t^{1/n}))$, where n is the degree of the extension. Conclude that it suffices to restrict the analysis to the case of division algebras D which are central simple. Let D have dimension n^2 over k, and consider a maximal commutative subfield L of D (of dimension n). Take an element $u \in L$ such that $u^n = t$, and find another element v such that $uv = \zeta vu$, $\zeta^n = 1$, and $v^n = f(t)$, so that we have a cyclic algebra. Derive that $n = 1$.

36. Show that if V is a generating subspace of an algebra A, and $f(n) = \mathrm{dim} V^n$, then

$$gk(A) = \mathrm{limsup}_{n \to \infty} \frac{\log f(n)}{\log n}.$$

37. Let G be the group of transformations of the line generated by $y = x + 1$ and $y = 2x$. Show that the group algebra of G over $\mathbb{Q}$ has exponential growth.

38. Classify irreducible representations of $U(sl(2))$ over an algebraically closed field of characteristic p.

39. Let k be an algebraically closed field of characteristic zero, and $q \in k^\times, q \neq \pm 1$. The quantum enveloping algebra $U_q(sl(2))$ is the algebra generated by e, f, K, K^{-1} with relations

$$KeK^{-1} = q^2 e, KfK^{-1} = q^{-2} f, [e, f] = \frac{K - K^{-1}}{q - q^{-1}}$$

(if you formally set $K = q^h$, you'll see that this algebra, in an appropriate sense, "degenerates" to $U(sl(2))$ as $q \to 1$). Classify irreducible representations of $U_q(sl(2))$. Consider separately the cases of q being a root of unity and q being not a root of unity.

40. Show that if R is a commutative unital ring, then a polynomial $p = a_0 + a_1 t + ... + a_n t^n$, $a_i \in R$, is invertible in $R[t]$ iff a_0 is invertible and a_i are nilpotent for $i > 0$.

Hint. Reduce to the case $a_0 = 1$. Then show that if p is nilpotent and $\chi : R \to K$ is a morphism from R to an algebraically closed field then $\chi(a_i) = 0$ for all i. Deduce that a_i are nilpotent.

41. (a) Show that $U(sl_2)$ is a PI algebra iff the ground field k has positive characteristic. What is the PI degree of this algebra? (smallest r such that the standard identity $S_{2r} = 0$ holds).

(b) For which q is the quantum group $U_q(sl_2)$ a PI algebra, and what is its PI degree?

42. Let K be an algebraically closed field of characteristic p ($p = 0$ or $p > 0$ is a prime). For $t, k \in K$, define the algebra $H_{t,k}$ over K generated by x, y, s with defining relations

$$sx = -xs, sy = -ys, s^2 = 1, [y, x] = t - ks$$

(the rational Cherednik algebra of rank 1). For which t, k, p is this a PI algebra, and what is its PI degree?