HOMEWORK 5 FOR 18.706, SPRING 2010
DUE WEDNESDAY, MAY 12.

Do 50% of the questions of your choice.

(1) (Adams operations in K-theory)

Let k be a field containing $\mathbb{Q}$ and all roots of unity.

(a) Let A be a k-linear abelian category and G be a finite group. Let A^G be the category of objects in A equipped with a G-action.
Define a map $\tau_A : K^0(A^G)_k \to K^0(A) \otimes_{\mathbb{Z}} k[G]^G$ where $k[G]^G$ is the space of k-valued class functions on G and $K^0(A^G)_k$ stands for $K^0(A^G) \otimes_{\mathbb{Z}} k$. The map τ_A should fit in the natural commutative diagram for a k-linear functor $A \to B$ and it should be an isomorphism when A is the category of finite dimensional k-vector spaces.

For $g \in G$ let $\tau_A^g : K^0(A^G)_k \to K^0(A)_k$ be the composition $(Id \otimes ev|_g) \circ \tau_A$ where $ev|_g : k[G]^G \to k$ is the map of evaluation on the conjugacy class of g.

(b) Let $R \supset k$ be a commutative ring and A be the category of finitely generated projective R-modules.[1]
For $M \in A$ the tensor product $M^{\otimes n} = M \otimes_A M \otimes \cdots \otimes_A M$ (where the number of factors in the RHS is n) carries an action of the symmetric group S_n by permutation of the factors. Thus $M^{\otimes n}$ acquires the structure of an object in $R - mod^{S_n}$.
Let $\sigma \in S_n$ be a long cycle.
Show that the $M \mapsto \tau^{\sigma}_{R-mod}([M^{\otimes n}])$ is additive on short exact sequences (unlike the map $M \mapsto [M^{\otimes n}]$). Thus it defines a homomorphism[2] $K^0(R - mod)_k \to K^0(R - mod)_k$.

(2) (Identities satisfied by characters)

(a) Let A be an algebra over a field k of characteristic zero, and $\chi : A \to k$ be the character (trace functional) of an n-dimensional representation.
Show that χ satisfies the following identity.
For $a_0, \dots, a_n \in A$ and $\sigma \in S_{n+1}$ set $\chi_\sigma(a_0, \dots, a_n) = \prod \chi(a_{i_1} a_{i_2} \dots a_{i_s})$ where $(i_1, \dots, i_s)$ runs over the cycles of σ.
Show that $\sum_{\sigma \in S_{n+1}} \epsilon(\sigma) \chi_\sigma(a_0, \dots, a_n) = 0$ for all $a_0, \dots, a_n \in A$; here $\epsilon(\sigma) = \pm 1$ depending on the parity of σ.
Write down the identity explicitly for $n = 1, 2$.

[1] This category A is not an abelian but rather an *exact* category. We don't discuss axiomatics of exact categories, it suffices to remark that the notion of an exact sequence in A has a clear meaning, thus the usual definition of K^0 applies to A, A^G.

[2] A similar definition applies when modules over a commutative ring R are replaced by vector bundles on a topological space, or algebraic vector bundles on an algebraic variety. The resulting endomorphisms of K^0 are known as *Adams operations*.

(b) Let G be a finite group. Show that a map $\chi : G \to k$ is a character of an n-dimensional isotypic representation iff it satisfies the following identity $\rho(g_1)\rho(g_2) = \frac{n}{|G|}\sum_{x\in G}\rho(xg_1x^{-1}g_2)$. (A representation is isotypic if it is a sum of several copies of the same irreducible representation.) [Hint: The key step is to show that the identity holds for irreducible characters. Write $Tr(g_1g_2) = Tr(g_1 \otimes g_2 \circ \sigma)$ where σ switches the factors in $V \otimes V$, then check that averaging σ under conjugations by elements of G acting on the first multiple gives $\frac{1}{n}Id$.]

(3) Prove that if $s \in R$ is regular and ad nilpotent then $GK\dim R[s^{-1}] = GK\dim(R)$.

(4) Check that the algebra described in problem 7,[3] pset 2, is a Koszul quadratic algebra with $A^! \cong A$.

(5) (Hochschild homology and cohomology.) Let A be an associative algebra over a field k.

The spaces $\mathrm{Ext}^i_{A\otimes A^{op}}(A, M)$ and $\mathrm{Tor}_i^{A\otimes A^{op}}(A, M)$, for a given A-bimodule M are called the Hochschild cohomology and homology spaces of A with coefficients in M, respectively, and denoted $HH^i(A, M)$ and $HH_i(A, M)$. [When M is not mentioned in the notation and/or wording, it is usually assumed that $M = A$ is the regular bimodule].

(a) Show that $HH^0(A)$ is the center of A, $HH_0(A) = A/[A, A]$ is the cocenter, $HH^1(A)$ is the space of derivations of A modulo inner derivations (i.e. commutators with an element of A).

(b) Show that for any A-modules M, N there is a natural action of the algebra $HH^*(A)$ on $Ext^*(M, N)$. More precisely, let $\mathcal{M}$ be a category defined as follows: objects of $\mathcal{M}$ are A-modules and $Hom_{\mathcal{M}}(M, N) = Ext^*(M, N)$. Then there exists a natural homomorphism from $HH^*(A)$ to the graded center of $\mathcal{M}$; i.e. HH^{even} maps to endomorphisms of $Id_{\mathcal{M}}$ while for n odd HH^n maps to $Ext^n(M, M)$ for all $M \in \mathcal{M}$ so that for all $h \in Ext^m(M, N)$ the natural diagram commutes if m is even and commutes after multiplication of one of the arrows by -1 otherwise.

(c) Let A_0 be an algebra over a field k. An n-th order deformation of A_0 is an associative algebra A over $k[t]/t^{n+1}$, free as a module over $k[t]/t^{n+1}$, together with an isomorphism of k-algebras $f : A/tA \to A_0$. Two such deformations (A, f) and (A', f') are said to be equivalent if there exists an algebra isomorphism $g : A \to A'$ such that $f'g = f$. Show that equivalence classes of first order deformations are parametrized by $HH^2(A_0, A_0)$.

(d) Show that if $HH^3(A_0) = 0$ then any n-th order deformation can be lifted to (i.e., is a quotient by t^{n+1} of) an $n + 1$-th order deformation.

(e) In 5e,5f k can be assumed to have characteristic zero.
Compute Hochschild cohomology of the polynomial algebra $A_0 = Sym(V)$ where V is a finite dimensional vector space over k. More precisely, show that it is isomorphic to the space of polynomial polyvector fields on the flat space V^*.
[Hint: do NOT use the bar complex]

[3]This algebra plays a role in representation theory as it controls the category of highest weight modules for the Lie algbera $sl(2)$.

(f) According to the above, a first order deformation of $A_0 = Sym(V)$ is determined by a bivector field $\alpha \in Sym(V) \otimes \wedge^2 V^*$. This bivector field defines a skew-symmetric bilinear binary operation on A_0, given by $\{f, g\} = (df \otimes dg)(\alpha)$. Show that the first order deformation defined by α lifts to a second order deformation if and only if this operation is a Lie bracket (satisfies the Jacobi identity). In this case α is said to be a Poisson bracket.

(g) Let $A = k\langle V\rangle/(I)$, $I \subset V \otimes V$ be a Koszul quadratic algebra. Then the homomorphism $HH^*(A) \to A^! = Ext^*_A(k, k)$ defined in 5b induces an isomorphism $\bigoplus\limits_n HH^n(A)_{(-n)} \to Z(A^!)$, while $HH^i(A)_{(j)} = 0$ for $j < -i$. Here Z stands for *supercenter*:

$$Z(A^!)^i = \{a \in (A^!)^i \mid ab = (-1)^{ij} ba \forall b \in (A^!)^j\}.$$

[Hint: Show that the i-th term in the minimal resolution for the regular bimodule over A has the form $((A^!)^i)^* \otimes (A \otimes A^{op})$ and identify a component of the differential in the corresponding complex for $HH^*(A)$ with the map $(A^!)^i \to V \otimes (A^!)^{i+1}$ obtained from the supercommutator map $V^* \otimes (A^!)^i \to (A^!)^{i+1}$ by "lowering the index".]

(h) Let $a \in Z((A^!))^2$ and let $h_a \in HH^2(A)$ be the corresponding element. Show that the first order deformation corresponding to h_a is isomorphic to the algebra over $k[t]/t^2$ with the space of generators V and relations $i - (a, i)t$, $i \in I$; here (a, i) is the pairing of $a \in (A^!)^2 = I^*$ and $i \in I$.

(i) $(*)$ Prove[4] that the deformation in the previous part is unobstructed (to all orders).

(6) (GK dimension does not behave well on short exact sequences)

Show that the the following provides an example of a PI algebra R, an R-module M with a submodule N, s.t. $GK \dim(N) = GK \dim(M/N) = 1$, $GK \dim(M) = 2$.

Set $R = \mathbb{C}\langle x, y\rangle/yx = 0$, let M have two generators α, β subject to relations: $x^{n+1} y^n \alpha = 0$ and $xy^n \beta = 0$ unless n is a square m^2 in which case $xy^n \beta = xy^m \alpha$. Let $N = R\beta$.

Check that R satisfies the identity $[a, b]^2 = 0$, thus it is PI.

[4] This proof may require some theory which was not discussed in class.