HOMEWORK 4 FOR 18.706, SPRING 2010
DUE WEDNESDAY, APRIL 21.

(1) Prove that a prime PI algebra with an identity of degree d has left (or right) uniform rank less than d.

(2) Let $A = k\langle x, y\rangle/xy - yx = 1$ be the Weyl algebra over an algebraically closed field of characteristic $p > 0$, and M be a finitely generated A-module of uniform rank d.

  (a) Assume that M is torsion free. Show that
$$\min_{a,b\in k}(\dim(M/(x^p - a, y^p - b))) = p^2 d.$$

  (b) Suppose that $y^p M = 0$ but M is torsion free over $k[x]$. Show that
$$\min_{a\in k}(\dim(M/(x^p - a))) = pd.$$

(3) Let R, S be rings where R is commutative. Suppose that $Mat_n(R) \cong Mat_n(S)$. Show that S is commutative and deduce that $R \cong S$.

[Hint: apply the standard identity for suitable arguments including ae_{11}, be_{11}.]

(4) Check that a $\mathbb{Q}$ algebra satisfying S_{2n+1} also satisfies S_{2n}.

(5) Classify irreducible (finite dimensional) representations of $U(sl(2), k)$ where k is algebraically closed of characteristic $p > 0$.

(6) Show that $U(sl(2), k)$ is a PI algebra iff the field k has positive characteristic.

(7) Let $\mathfrak{g} = gl(n, \mathbb{Q})$, $G = GL(n, \mathbb{Q})$. Define $\sigma_k : \Lambda^{2k-1}(\mathfrak{g}) \to \mathbb{Q}$ by
$$x_1 \wedge x_2 \cdots \wedge x_{2k-1} \mapsto \sum_s (-1)^{|s|} Tr(x_{s(1)} \ldots x_{s(2k-1)}).$$

  (a) Show that $\sigma_k = 0$ iff $k > n$.

  (b) $(*)$ Prove[1] that the algebra $\Lambda(\mathfrak{g}^*)^G$ is a free graded commutative algebra generated by $\sigma_1, \ldots, \sigma_n$.

[1] A standard short proof relies on the theory identifying $\Lambda(\mathfrak{g}^*)^G \otimes_{\mathbb{Q}} \mathbb{C}$ with topological cohomology $H^*(U(n), \mathbb{C})$ where $U(n)$ is the unitary group considered as a topological space. A purely algebraic proof would be more befitting this course, but it may be more tedious to find.