HOMEWORK 3 FOR 18.706, SPRING 2010
DUE WEDNESDAY, MARCH 31.

(1) An element x in a ring R is said to be ad locally nilpotent if $ad(x) : a \mapsto xa - ax$ is locally nilpotent. Show that a multiplicative set consisting of ad locally nilpotent elements satisfies Ore's condition.

(2) Let K be a skew field and $\phi : K \to K$ a homomorphism. Set $A = K\langle\phi\rangle/(xa = \phi(a)x)$.

Show that the set of powers of x is a left Ore set, and it is a right Ore set iff ϕ is surjective. In the latter case prove that A is a right Ore domain, i.e. the set of all nonzero elements is right localizing.

(3) Construct an embedding of the free ring $\mathbb{C}\langle x, y\rangle$ into a skew field.

[Hint: use the previous problem and Jategaonkar's lemma]

(4) Give an example of a rank 4 non-split Azumaya algebra over the ring of real valued continuous functions on the 2-sphere S^2 such that for every $s \in S_2$ the c.s.a. over $\mathbb{R}$ induced by means of the evalutation homomorphism at s is split.

[Basic topology can be used here]

(5) Let R be a discrete valutation ring with fraction field K and A be an Azumaya algebra over R. Show that if the c.s.a. $A \otimes_R K$ is split, then A is split.

(6) Let R be a ring with elements a, b such that $ab = 1 \neq ba$. Set $e_{11} = 1 - ba$, $e_{ij} = b^{i-1}e_{11}a^{j-1}$. Show that $e_{ij}e_{kl} = \delta_{jk}e_{il}$. Deduce that R has infinite uniform rank.

(7) Let us say that an ideal $I \subset R$ is right localizable if the set of elements regular modulo I is a right Ore set.

Let k be a field and $R \subset Mat_2(k[x])$ be given by $R = \{(a_{ij}) \mid a_{21} = 0,\ a_{11} \in k,\ a_{22} - a_{11} \in xk[x]\}$. Show that R is right Noetherian, the ideal of strictly upper triangular matrices is prime, has square zero and is not localizable.

(8) Show that if R is an Ore domain, i.e. the set of all nonzero elements is both left and right localizing, then every finitely generated flat module is projective.

(9) Let F be the ring of continuous functions on $[0, 1]$ with pointwise operations. Show that F has no uniform ideals.