HOMEWORK 2 FOR 18.706, SPRING 2010
DUE WEDNESDAY, MARCH 10.

(1) Let M be an indecomposable module of finite length over some ring R. Show that $End(M)$ has a unique irreducible module. If R is a k-algebra for an algebraically closed field k and M is finite dimensional over k, show that the only irreducible representation of $End(M)$ is one dimensional over k.

(2) Let R be a ring and F be the forgetful functor from the category of R-modules to abelian groups. Describe $End(F)$.

More generally, let $A \to R$ be a ring homomorphism. Describe the endomorphism ring of the pull-back functor $R-mod \to A-mod$.

(3) Show that any derivation of a central simple algebra is inner.

(4) Prove that if D is a central division algebra over a perfect field k of characteristic p then $\dim_k(D)$ is prime to p. Deduce that the Brauer group of k has no p-torsion.

(5) Let D be the division algebra over $K = \mathbb{F}_p(t_1, t_2)$ introduced in problem 7 of pset 1.

(a) Find an explicit separable splitting field of degree p for D.

(b) Let us generalize the definition of D as follows. For $P \in \mathbb{F}_p[t]$ let $D(P)$ be given by $x^p = P(t_1)$, $y_p = t_2$, $xy - yx = 1$.

Check that $P \to [D(P)]$ is a homomorphism from the additive group of polynomials to $Br(K)$.

(c) Show that the kernel of this homomorphism is $\mathbb{F}_p[t^p]$. Conclude that p-torsion in the Brauer group of K is infinite.

(6) For a ring A its *cocenter* is $C(A) = A/[A, A]$ where $[A, A]$ denotes the additive subgroup in A generated by commutators $xy - yx$.

(a) Check that $C(A) = A \otimes_B A$ where $B = A \otimes A^{op}$.

(b) Consider the abelian group $\tau(A)$ generated by pairs (P, r) where P is a finitely generated projective A-module and r is an endomorphism of P subject to the relations $(P, \phi + \psi) \sim (P, \phi) + (P, \psi)$, $(P, gf) \sim (Q, fg)$ for two modules P, Q and $f : P \to Q$, $g : Q \to P$, $\phi, \psi : P \to P$. Show that the map sending a to (A, r_a) where $r_a : x \to xa$ induces an isomorphism $A/[A, A] \to \tau(A)$. The inverse map sends (P, r), where $A^n = P \oplus Q$ and $r \oplus 0$ is given by a matrix r_{ij}, to $\sum r_{ii} \mod [A, A]$. [The map sending (P, r) to the corresponding element in $A/[A, A]$ is sometimes called *Hattori-Stallings trace*].

(c) Suppose that A is Noetherian of finite homological dimension; thus every finitely generated module has a finite resolution by finitely generated projective modules. Prove that $C(A)$ is also isomorphic to the abelian group generated by pairs (M, r) where M is a finitely generated (not necessarily projective) module and r is its endomorphism, subject to the above relations together with the following additional relation: for an exact sequence $0 \to M_1 \to M_2 \to M_3 \to 0$ and endomorphisms f_i of M_i $(i = 1, 2, 3)$ fitting into a commutative diagramm we impose the relation $[M_2, f_2] = [M_1, f_1] + [M_3, f_3]$.

(d) Suppose that A is finite dimensional over an algebraically closed field k. Prove that if A has finite homological dimension, then characters of irreducible representations form a basis in $C(A)^*$.

(e) Prove that a commutative finite dimensional k algebra which has finite homological dimension is a product of several copies of k.

(7) Show that the following ring A is finite dimensional over $\mathbb{C}$ and has finite homological dimension. Let Q be the quiver with two vertices and two edges connecting them, the two edges have opposite orientation. Set $A = A(Q)/(e_1e_2)$ where e_1, e_2 are the elements corresponding to the edges.

[It is a standard fact that an Artinian ring has finite homological dimension provided that each irreducible module has a finite projective resolution. You can use this fact.]