HOMEWORK 1 FOR 18.706, FALL 2010
DUE MONDAY, FEBRUARY 22.

(1) Let R be the subring in $Mat_2(\mathbb{R})$ given by $R = \{(a_{ij}) \mid a_{21} = 0,\, a_{22} \in \mathbb{Q}\}$. Show that R is left Artinian but not right Artinian.

(2) Let us say that a ring R is *rankable* if the free modules $R^{\oplus n}$, $R^{\oplus m}$ are not isomorphic for $m \neq n$.
   (a) Let $R_1 \to R_2$ be a (unital) homomorphism. Show that if R_2 is rankable that so is R_1.
   (b) Prove that a left Noetherian ring is rankable.
   (Hint: it may be a good idea to postpone doing this problem until we discuss Goldie's Theorem.)
   (c) Let k be a field, V be an infinite (countably) dimensional vector space over k. Set $R = End_k(V)$. Check that R is not rankable by establishing an isomorphism between the free R modules of ranks one and two. Conclude that R does not admit homomorphisms to $Mat_n(D)$ where D is a division ring.

(3) A module homomorphism $M \to N$ is called *essential* if it is surjective but its restriction to any proper submodule in M is not surjective. A *projective cover* of a module N is an essential homomorphism from a projective module.
   (a) Show that a projective cover of a given module is unique up to a non-canonical isomorphism if it exists.
   (b) Let R be an Artinian ring. Show that a non-zero homomorphism from a projective R-module P to an irreducible module L is essential iff P is indecomposable.
   Deduce that every irreducible R module admits a projective cover.
   (c) Let $L_1, \dots, L_n$ be a set of representatives for isomorphism classes of irreducible R-modules.
   Let P_i be a projective cover of L_i. Show that any finitely generated projective module is of the form $\sum P_i^{\oplus d_i}$. Express d_i in terms of $Hom(P, L_i)$.
   (d) Assume that R is a finite dimensional algebra over an algebraically closed field k. Set $m_i = \dim(L_i)$, $p_{ij} = \dim Hom(P_i, P_j)$. (The matrix p_{ij} is called Cartan matrix of R). Express $\dim(R)$, $\dim(P_i)$ in terms of m_i, p_{ij}.
   Give an example showing the formula you obtained does not necessarily hold if k is not required to be algebraically closed.
   (e) Find m_i, p_{ij} for R being the ring of upper triangular matrices over a field k.

(4) Let Q be a quiver, i.e. a finite oriented graph. Let $A(Q)$ be the path algebra of Q over a field k, i.e. the algebra whose basis is formed by paths in Q (compatible with orientations, and including paths of length 0 from a

vertex to itself), and multiplication is concatenation of paths (if the paths cannot be concatenated, the product is zero).

(a) Represent the algebra of upper triangular matrices as $A(Q)$.
(b) Show that $A(Q)$ is finite dimensional iff Q is acyclic, i.e. has no oriented cycles.
(c) For any acyclic Q, decompose $A(Q)$ (as a left module) in a direct sum of indecomposable modules, and classify the simple $A(Q)$-modules.
(d) Find a condition on Q under which $A(Q)$ is isomorphic to $A(Q)^{op}$, the algebra $A(Q)$ with opposite multiplication. Use this to give an example of an algebra A that is not isomorphic to A^{op}.

(5) This problem provides examples showing that the conclusion of the Krull-Schmidt Theorem does not hold without the finiteness assumption on the module.
(a) Let A be the algebra of smooth real functions on the real line, such that $a(x+1) = a(x)$. Let M be the A-module of smooth functions on the line such that $b(x+1) = -b(x)$.
Show that M is indecomposable and not isomorphic to A, and that $M \oplus M = A \oplus A$ as a left A-module. Thus the conclusion of Krull-Schmidt theorem does not hold in this case (the theorem fails because the modules we consider are infinite dimensional).
(b) Let R be a Dedekind domain[1] which is not a principal ideal domain (e.g. $R = \mathbb{Z}[\sqrt{-5}]$). Show that the conclusion of Krull-Schmidt Theorem does not hold for finitely generated projective R modules.
(Hint: you can use that $I \oplus J \cong R \oplus IJ$ for nonzero ideals $I, J \subset R$).

(6) Let D be a noncommutative division algebra over a field k of characteristic p. Suppose that $\dim_k(D) = p^2$. Show that the character of any finite dimensional representation of D vanishes.

(7) Show that the following provides an example of the situation described in the previous problem. Let $k = \mathbb{F}_p(t_1, t_2)$ be the field of rational functions in two variables over $\mathbb{F}_p$. Set $D = k\langle x, y\rangle/(xy - yx = 1,\, x^p = t_1,\, y^p = t_2)$.

(Hint: Think of D as the Weyl algebra $k\langle x, y\rangle/(xy - yx = 1)$ localized at polynomials in central elements x^p, y^p. Check that D is a central simple algebra with no zero divisors.)

[1]Recall that this means that R is a Noetherian (commutative) domain where every ideal is a product of prime ideals. Rings of integers in number fields provide important examples of Dedekind domains.