HOMEWORK 4 FOR 18.706, SPRING 2012
DUE WEDNESDAY, APRIL 11.

(1) Let k be a field of characteristic zero. Recall the Weyl algebra

$$\mathbf{W}_n = k\langle x_1, \dots, x_n, y_1, \dots y_n\rangle / (y_i x_j - x_j y_i - \delta_{ij}, y_i y_j - y_j y_i, x_i x_j - x_j x_i).$$

The algebra $\mathbf{W}_n$ acts on the space of polynomial in n variables $P_n = k[t_1, \dots, t_n]$ so that x_i acts by multiplication by t_i and y_i acts by $\frac{d}{dx_i}$.

(a) Check that P_n is simple.

(b) [1] Compute $Ext_{\mathbf{W}_n}(P_n, P_n)$.
[Hint: Do the case $n = 1$ first, use an explicit projective resolution of L. The tensor product of n copies of this gives a resolution of L for any n.
An alternative approach is to observe that the complex for computation of $Ext_{\mathbf{W}_n}(P_n, P_n)$ coming from the natural projective resolution for P_n is the Koszul complex for exterior algebra, also known as De Rham complex for $\mathbb{A}^n$.]

(c) [2] Let $\mathcal{A} \subset \mathbf{W}_2 - mod$ be the subcategory of modules on which x_1, x_2 act locally nilpotently. Let $\bar{P}_2$ be the image of P_2 in the Serre quotient $\mathbf{W}_2 - mod/\mathcal{A}$. Compute $Ext(\bar{P}_2, \bar{P}_2)$.
[Hint: if $M \in \mathbf{W}_2 - mod$ is such that either x_1 or x_2 acts on M invertibly, then $M \in \mathcal{A}^\perp$, and it follows that $Ext^i(X, M) \cong Ext^i(\bar{X}, \bar{M})$ for all X, i, in the self-explanatory notation. Find a right resolution of P_2 by such modules.]

(2) [3] Let A with be the four dimensional ring over $\mathbb{C}$ such that $A = \mathbb{C}e_1 \oplus \mathbb{C}e_2 \oplus I$ where e_1, e_2 are orthogonal idempotents, $e_1 I = I = I e_2$ and $I e_1 = 0 = e_2 I$. Compute Hochschild homology of A.

(3) Let R the ring of real valued continuous functions on the 2-sphere S^2. Let R^+ and R^- be the ring of continuous functions on the upper and lower closed hemispheres respectively. Let $A \subset Mat_2(R^+) \oplus Mat_2(R^-)$ be the subring given by $(m_+, m_-) \in A$ if $m_+(\theta) = S(\theta) m_-(\theta) S(\theta)^{-1}$. Here $\theta \in [0, 2\pi)$ is the standard coordinate on the equator circle bounding the upper and the lower hemisphere, and

$$S(\theta) = \begin{pmatrix} \cos(\frac{\theta}{2}) & \sin(\frac{\theta}{2}) \\ -\sin(\frac{\theta}{2}) & \cos(\frac{\theta}{2}) \end{pmatrix}$$

Prove that A is a non-split Azumaya algebra over R.

[Hint: Basic topology can be used in this problem. Reduce the statement to the fact that the map $\pi_1(S^1) \to \pi_1(S^1)$ induced by the double cover map $S^1 \to S^1$ is not surjective.]

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[1] This is an algebraist's way to compute cohomology of Euclidean space $\mathbb{C}^n \cong \mathbb{R}^{2n}$.

[2] This is an algebraist's way to compute cohomology of $\mathbb{C}^2 \setminus 0 \sim S^3$.

[3] The answer in this problem should coincide with the answer for $H^*(\mathbb{CP}^1, \mathbb{C})$ put all in degree zero. This can be derived from Hodge theory, derived Morita invariance of Hochschild homology and Hochschild-Kostant-Rosenberg isomorphism.

(4) Let k be a field, $F = k((t))$, $R = k[[t]]$.

(a) Let A be an Azumaya algebra over R. Show that the following are equivalent

(i) A is split.

(ii) $A \otimes_R k$ is split.

(iii) $A \otimes F$ is split.

[Hint: for (ii) $\Rightarrow$ (i) show (or recall) that an idempotent in a ring S lifts an idempotent in $\tilde{S}$ if $S = \tilde{S}/I$, $I^2 = 0$. For (iii) $\Rightarrow$ (i) show that a finitely generated projective module P_F for $A \otimes F$ is obtained from a finitely generated projective module P for A by extension of scalars – just choose generators for P_F arbitrarily, let P be the A submodule generated by these, and check that P is projective].

(b) Prove that every Azumaya algebra over the ring of p-adic numbers $\mathbb{Z}_p$ splits.

(c) (Optional) Let $\mathcal{A}$ be a central simple algebra over F. Show that $F \cong A \otimes_R F$ for some Azumaya algebra A over R iff the determinant of the pairing $(a, b) \mapsto \tau(ab)$ is an invertible element of R. Here τ is the reduced trace.

[Hint: reduce to the case when A is a division ring. In this case show that the subset of elements whose determinant is in R forms a subring, and this subring is simple provided that the determinant of the pairing is in $R^\times$].