HOMEWORK 3 FOR 18.706, SPRING 2012
DUE WEDNESDAY, MARCH 21.

(1) Let $\mathcal{A}$ be the category of finitely generated modules over $\mathbb{C}[x]$.
  (a) Prove that $\mathcal{A}^{op}$ is not equivalent to the category of finitely generated modules over any algebra.
  (b) Give an example of a Serre subcategory $\mathcal{B} \subset \mathcal{A}$, so that both $\mathcal{B}$ and $\mathcal{A}/\mathcal{B}$ are equivalent to its opposite category.

(2) Let $\mathcal{A}$ be the category of vector spaces over a field k which are at most countably dimensional, and let $\mathcal{B}$ be the Serre subcategory of finite dimensional vector spaces. Prove that the set of isomorphism classes of objects in $\mathcal{A}/\mathcal{B}$ has two elements. Show that the algebra of endomorphisms of a nonzero object in $\mathcal{A}/\mathcal{B}$ is simple.

(3) Let Γ be a finite group acting on a ring R by automorphisms. Then the *smash product*[1] $\Gamma\#R$ or $\Gamma \ltimes R$ is the abelian group $\bigoplus_{\gamma\in\Gamma} R$ with multiplication given by: $(r_{\gamma_1})(r'_{\gamma_2}) = r\gamma_1(r')_{\gamma_1\gamma_2}$ in the self-explanatory notation.

  Suppose that R is simple and $|G|$ is invertible in R.

  Suppose that one the following two conditions holds.
  (a) No nontrivial element $\gamma \in \Gamma$ acts by an inner automorphism of R.
  (b) The element $\gamma \in \Gamma$ acts by conjugation by an invertible element $r_\gamma \in R$, where the elements $r_\gamma \in R$ are linearly independent over the center of R.

  Prove that the functor $M \mapsto M^\Gamma$ is an equivalence between the category of $\Gamma\#R$-modules and of R^Γ-modules.

(4) Let k be a field and define the Weyl algebra
  $\mathbf{W}_n = k\langle x_1, \ldots, x_n, y_1, \ldots y_n\rangle/(y_ix_j - x_jy_i - \delta_{ij}, y_iy_j - y_jy_i, x_ix_j - x_jx_i)$.
  (a) Prove that monomials x^Iy^J form a basis of $\mathbf{W}_n$.
  (b) Prove that $\mathbf{W}_n$ is simple iff k is of characteristic zero.
  (c) Define a natural action of $Sp(2n, k)$ on $\mathbf{W}_n$. For a finite subgroup $\Gamma \subset Sp(2n)$ show that $\mathbf{W}_n^\Gamma$ is Morita equivalent to $\Gamma\#\mathbf{W}_n$ if k is of characteristic zero.
  (d) Suppose that $\Gamma \subset Sp(2n, \mathbb{Z})$, i.e. Γ acts on a lattice preserving a nondegenerate skew-form. For a field k consider the composed map $\Gamma \to Sp(2n, \mathbb{Z}) \to Sp(2n, k)$ where the second arrow comes from tensoring the lattice by k. Show that for almost all p we have Morita equivalence $\mathbf{W}_{n,k}^\Gamma \sim \Gamma\#\mathbf{W}_{n,k}$ for all fields k of characteristic p.
  (e) (Optional) Consider the standard skew form on $\mathbb{Z}^{2n}$ given by a block diagonal matrix of n identical 2×2 blocks ($\Omega_{2i,2i-1} = 1 = -\Omega_{2i-1,2i}$ for $i = 1, .., n$, and the other entries are zero). Let $\Gamma = S_n$ acting on $\mathbb{Z}^{2n}$ by permuting the blocks. Find an explicit bound on p satisfying the condition in the previous question.

[1] I prefer the notation which can be described as "semi-tensor product" but I don't know how to reproduce it in LaTeX!

(5) Let $A = \bigoplus_{n=0}^{\infty} A_n$ be a graded algebra over a field k with finite dimensional graded component. Define the Cartan matrix $C \in Mat_d(Z[[t]])$ by $C_{i,j} = \sum t^n \dim Hom(P_i, P_j(n))$. Here d is the number of isomorphism classes of irreducible modules concentrated in graded degree zero, P_i are the projective covers of those irreducibles and $M(n)$ for a graded module M denotes the module M with shifted grading: $M_m(n) = M_{m+n}$.

Assume that A is of finite homological dimension. Prove that $\det(C)$ is the Taylor series of a nonzero rational function in t.