HOMEWORK 2 FOR 18.706, FALL 2012
DUE WEDNESDAY, MARCH 7.

(1) Let R be a ring and F be the forgetful functor from the category of R-modules to abelian groups. Describe $End(F)$.

More generally, let $A \to R$ be a ring homomorphism. Describe the endomorphism ring of the pull-back functor $R - mod \to A - mod$.

(2) For each of the following functors determine existence of a left adjoint, of a right adjoint and describe the existing adjoint functors.

(a) Let $Q = (\bullet \longrightarrow \bullet)$ be the quiver with two vertices and one arrow between them. Let $\mathcal{B}$ be the category of representations of Q over a fixed field, $\mathcal{A}$ be the full subcategory consisting of such representations that the map between the two vector spaces is injective, and $F : \mathcal{A} \to \mathcal{B}$ be the embedding.

(b) A graded commutative (or *super-commutative*) ring is a $\mathbb{Z}/2\mathbb{Z}$ graded ring $R = R_0 \oplus R_1$ such that $xy = -yx$ for $x, y \in R_1$ and $xy = yx$ for $x \in R_0$, $y \in R$.

Let $\mathcal{A}$ be the category of vector spaces over a field k not of characteristic two, $\mathcal{B}$ be the category of graded commutative k-algebras and let F send a vector space to its exterior algebra.

(3) Show that the category Ab_{gr} of $\mathbb{Z}$-graded abelian groups is complete (has arbitrary products), and that the functor of forgetting the grading $Ab_{gr} \to Ab$ does not commute with infinite products.

(4) An Artin ring is called self-injective if the free module is injective. A finite dimensional algebra A over a field k is called Frobenius if there exists a linear functional τ on A such that the bilinear pairing $A \times A \to k$, $(x, y) \mapsto \tau(xy)$ is non-degenerate.[1]

(a) Prove that a Frobenius algebra is self-injective.

(b) Let A be a finite dimensional algebra over a field. Assume that for every simple A-module L the multiplicity of L in the co-socle of A viewed as a left A-module equals the multiplicity of L^* in the socle of A viewed as a right A-module. Show that A is self-injective.

(Here we use that for a left A-module M the dual vector space $M^* = Hom(M, k)$ carries a right A-module structure, the action is given by the adjoint operators).

(5) Show that the following ring A is finite dimensional over $\mathbb{C}$ and has finite homological dimension. Let Q be the quiver with two vertices and two edges of opposite orientation connecting them. Set $A = A(Q)/(e_1 e_2)$ where e_1, e_2 are the elements corresponding to the edges.

[Hint: Recall that an Artinian ring has finite homological dimension provided that each irreducible module has a finite projective resolution.]

[1] It is easy to see that the group algebra $k[G]$ of a finite group G and exterior algebra of a finite dimensional vector space are examples of Frobenius algebras. Another class of examples is provided by cohomology of compact oriented manifolds, this follows from Poincare duality.