HOMEWORK 9 FOR 18.745, FALL 2012
DUE FRIDAY, NOVEMBER 9 BY 3PM.

The base field is algebraically closed of characteristic zero and Lie algebras and their representations are finite dimensional unless stated otherwise.

(1) For a semi-simple Lie algebra $\mathfrak{g}$ let G be the group of automorphisms of $\mathfrak{g}$ generated by elements of the form $\exp(ad(x))$, $x \in \mathcal{N}$, where $\mathcal{N} \subset \mathfrak{g}$ is the set of nilpotent element.

Let $\mathfrak{g} = \mathfrak{sl}_n$ and let the automorphism ι be given by $\iota(x) \mapsto -x^t$, where x^t denotes the transposed matrix. Show that $\iota \in G$ iff $n = 2$.

[Hint: $\iota \notin G$ for $n > 2$ can be deduced from the fact that the tautological n-dimensional representation of $\mathfrak{sl}_n$ is not isomorphic to its dual].

(2) [1] Let W be the Weyl group of a root system $\Sigma \subset E$.

   (a) Show that $-Id_E \in W$ iff Σ contains an orthogonal basis of E.
   [Hint: use that for any vector in $v \in E$ the stabilizer $Stab_W(v)$ is generated by reflections at root hyperplanes for roots orthogonal to v. Then apply induction in $\dim(V)$.]

   (b) For which classical algebras do the equivalent conditions from (a) hold?

(3) It is easy to see that for a non-simply root system the set of long roots also forms a root system.

   (a) For each non-simply laced irreducible root system of classical type determine the isomorphism class of the root system formed by long roots in Σ.

   (b) (Optional) Do the same for all irreducible non-simply laced root systems.

[1]This problem is related to the previous one in the following way: in fact, the equivalent conditions in (a) hold for the root system of a semi-simple Lie algebra $\mathfrak{g}$ iff every representation of $\mathfrak{g}$ is isomorphic to its dual.