HOMEWORK 8 FOR 18.745, FALL 2012
DUE FRIDAY, NOVEMBER 2 BY 3PM.

The base field is algebraically closed of characteristic zero and the Lie algebras are finite dimensional unless stated otherwise.

(1) Let $\mathfrak{t}$ be a maximal toral subalgebra in a semisimple Lie algebra $\mathfrak{g}$. Let $\Sigma = \Sigma(\mathfrak{g}) = \{\alpha \in \mathfrak{t}^* | \ \alpha \neq 0 \ \& \mathfrak{g}_\alpha \neq 0\}$, so that, as has been shown in class, $\Sigma \subset E := \mathbb{Q} \cdot \Sigma \otimes_{\mathbb{Q}} \mathbb{R}$ is a root system, where the form on E comes from the Killing form on $\mathfrak{g}$.

 (a) Recall that for every $\alpha \in \Sigma$ we have an $\mathfrak{sl}_2$ triple $e_\alpha \in \mathfrak{g}_\alpha$, $f_\alpha \in g_{-\alpha}$, $h_\alpha \in \mathfrak{t}$. Set $\Sigma' = \{h_\alpha \mid \alpha \in \Sigma\}$. Show that $\Sigma' \subset \mathbb{Q} \cdot \Sigma' \otimes_{\mathbb{Q}} \mathbb{R} \cong E^*$ is also a root system.

 (b) Let $\mathfrak{g} = \mathfrak{sp}_6$. The root system Σ' is not isomorphic to Σ, but it is isomorphic to $\Sigma(\mathfrak{g}')$ for another semi-simple Lie algebra $\mathfrak{g}'$. Describe $\mathfrak{g}'$ and prove this statement.
 [Here we say that two irreducible root systems $\Sigma_1 \subset E_1$, $\Sigma_2 \subset E_2$ are isomorphic if there exists an isomorphism $\phi : E_1 \cong E_2$ sending Σ_1 to Σ_2 and satisfying $(\phi(x), \phi(y)) = C(x, y)$ for some constant c].
 [Hint: You may want to check that Σ' is isomorphic to the root system $\check{\Sigma} = \{\frac{\alpha}{(\alpha,\alpha)} \mid \alpha \in \Sigma\}$.

(2) Check that for any nontrivial representation of $\mathfrak{sl}_2$ the corresponding trace form is nondegenerate.

(3) Let $\mathfrak{g}$ be a semi-simple Lie algebra and $x \in \mathfrak{g}$ be a semi-simple element. Show that the centralizer of x is a reductive Lie algebra.
 [Hint: Let $\mathfrak{t}$ be a maximal toral subalgebra containing x. Then the centralizer is the sum of $\mathfrak{t}$ and $\mathfrak{g}_\alpha$ for α such that $\alpha(x) = 0$. Recall Cartan criterion and use the previous problem to show that the kernel of the Killing form of the centralizer is contained in the subspace of $y \in \mathfrak{t}$, s.t. $\alpha(y) = 0$ for α as above.]

(4) (Optional) Let $\mathfrak{g}$ be any Lie algebra.

 (a) Show that the tensor product of two semi-simple $\mathfrak{g}$ modules is semi-simple.

 (b) Let $\mathfrak{k}$ be the intersection of kernels of all irreducible representations of $\mathfrak{g}$. Show that $\mathfrak{g}/\mathfrak{k}$ is reductive, and that it is a maximal reductive quotient of $\mathfrak{g}$ (i.e. for any ideal $\mathfrak{k}'$ in $\mathfrak{g}$ such that $\mathfrak{g}/\mathfrak{k}'$ is reductive we have: $\mathfrak{k} \subset \mathfrak{k}'$).
 [Hint: use the optional problem from the previous problem set.]

(5) (Optional) Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{R}$. Show that the Killing form of $\mathfrak{g}$ can not be positive definite.[1]
 [Hint: Pick $x \in \mathfrak{g}$ and consider the eigenvalues of $ad(x)$ acting on $\mathfrak{g} \otimes \mathbb{C}$. If there exists an eigenvalue with a nonzero real part deduce that $\mathfrak{g}$ contains and ad nilpotent element y, then $\kappa(y, y) = 0$. Otherwise all eigenvalues of $ad(x)$ are purely imaginary, so $kappa(x, x) \leq 0$.]

[1] It may however be negative definite.