HOMEWORK 7 FOR 18.745, FALL 2012
DUE FRIDAY, OCTOBER 26 BY 3PM.

The base field is algebraically closed of characteristic zero and the Lie algebras are finite dimensional unless stated otherwise.

(1) Let $C \in U(\mathfrak{sl}_n)$ be the Casimir element corresponding to the Killing form. How does C act in the tautological representation $V = k^n$?

(2) (Higher Casimir elements)[1]

(a) (Optional) Fix $d > 0$. Show that the element $C_d \in U(\mathfrak{gl}_n)$ given by:

$$C_d = \sum_{i_1,\ldots,i_d} E_{i_1 i_2} E_{i_2 i_3} \cdots E_{i_d i_1}$$

is central.

[Hint: Degree d symmetric tensors can be identified with polynomial functions on the dual space. Identifying $\mathfrak{g} = \mathfrak{gl}_n$ with $\mathfrak{g}^*$ by means of the trace form, C_d gets identified with the polynomial $A \mapsto Tr(A^d)$]

(b) Let $V = k^3$ be the tautological representation of $\mathfrak{sl}_3$, extend it to a representation of $\mathfrak{gl}_3$ so that scalar matrices act by zero. Show that C_2 acts by the same scalar on V and on V^*, while C_3 acts on these two representations by different scalars. Compute these scalars.

(3) Let $e \in sl_n$ be the matrix with $e_{i,i+1} = 1$, $i = 1, \ldots, n-1$ and $e_{i,j} = 0$ for $j \neq i + 1$.

(a) Show that e can be uniquely completed to an $\mathfrak{sl}_2$ triple (e, h, f) so that h is diagonal. Compute h, f.

(b) Find a basis in the centralizer of e in $\mathfrak{sl}_n$ and describe the isomorphism class of $\mathfrak{sl}_n$ as a module over $\mathfrak{sl}_2$ acting by the adjoint representation restricted to the span of e, h, f (i.e. find $n_1, \ldots, n_k$ so that $\mathfrak{sl}_n \cong V_{n_1} \oplus \cdots \oplus V_{n_k}$).

(c) Let $n = 3$, and let $h' = h + n$ where $n_{i,j} = 1$ for $i < j$ and $n_{i,j} = 0$ for $i \geq j$. Find an invertible matrix g such that $gh'g^{-1} = h$.

[You can give the answer as a product of exponents of nilpotent matrices, you don't have to evaluate the product].

(4) (Optional) Show that for any Lie algebra $\mathfrak{g}$ the radical of $\mathfrak{g}$ acts by a scalar operator in any irreducible representation of $\mathfrak{g}$.

[Hint: Let $\mathfrak{r}$ be the radical of $\mathfrak{g}$ and V an irreducible representation. The key step is to show that $[\mathfrak{g}, \mathfrak{r}]$ acts on V by zero. Deduce from Lie Theorem that there exists a vector in V which is an eigenvector for all $x \in \mathfrak{r}$, on this vector $\mathfrak{r}'$ acts by zero; hence it acts by zero on the subrepresentation generated by v, which is the whole V. Now we can assume that $\mathfrak{r}$ is abelian; check then that $\mathfrak{g}$ preserves generalized eigenspaces of each element in $\mathfrak{r}$.

[1] This problem illustrates the following point. For $\mathfrak{g} = \mathfrak{sl}_2$ we have proven splitting of a short exact sequence $0 \to L_1 \to M \to L_2 \to 0$ where L_1, L_2 are nonisomorphic irreducible modules by observing that Casimir element acts on L_1, L_2 by distinct scalars. For other semi-simple Lie algebras the Casimir element may act by the same scalar on two given representations L_1, L_2, but one can always find some central element in the enveloping algebra which acts by distinct ones.

Since V is irreducible, each element in $\mathfrak{r}$ has only one eigenvalue. Considering trace conclude that each element in $[\mathfrak{g}, \mathfrak{r}]$ acts nilpotently. By Engel's Theorem $[\mathfrak{g}, \mathfrak{r}]$ kills some subspace in V, which has to be invariant under $\mathfrak{g}$, hence coincides with V. Now we know that $[\mathfrak{g}, \mathfrak{r}]$ acts by zero, and the statement follows from Schur Lemma.]